Aim high and go far—Optimal projectile launch angles greater than 45°

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(Received 27 February 1997; accepted 21 October 1997)

For ideal projectile motion, which starts and ends at the same height, maximum range is achieved when the firing angle is 45° . If air resistance is taken into account, the optimal angle is somewhat less than 45° and this is often considered obvious. We show here that it is not obvious by considering drag forces with different dependence on projectile speed. In some cases maximum range is achieved for launch angles greater than 45° . Simple physical arguments are given which help explain results that were found by computing trajectories and ranges. © 1998 American Association of Physics Teachers.

If mechanics is food for thought, then projectile motion provides much of the starch at the beginning of introductory physics courses. One of the more interesting results served is the relationship of the range (horizontal distance traveled) of a projectile and the angle θ of inclination at its launch. For a fixed initial speed, and for negligible air resistance, the range is a function of θ that is symmetric about 45°. The range for 30°, for example, is the same as the range for 60°. The optimal angle, i.e., the angle for maximum range, is 45°.

A more difficult question is what the optimal angle is if air resistance is taken into account.¹ It is semicommon knowledge that maximum range, when the resistance of still air is taken into account, occurs for angles less than 45°.² But is this answer obvious? An informal survey of our colleagues revealed that the immediate reaction is "yes, it's obvious." This seems to be related to the fact that a lower trajectory reduces the time and distance of flight, therefore minimizing the time and distance over which the drag force due to the air is acting. This point, in fact, is explicitly made in one of the texts.³ Another introductory textbook⁴ clearly treats the issue as obvious, since the question is assigned to the students at the end of the chapter on two-dimensional motion. At this point in the text. Newton's laws have not even been introduced, let alone a description of air resistance. The implication is that the decrease in the optimal launching angle is a robust result, independent of the details of the resistive force. This, in fact, is precisely what seems to be proved in a recent publication in this Journal,⁵ i.e., that for any force directed opposite to the velocity of the projectile, the optimal angle cannot be greater than 45°. The proof, however, suffers from a flaw in the formulation of the problem,⁶ so the question remains open and interesting.

One can get a first insight into the nature of the problem by considering 30° and 60° launches, cases that have equal range in the absence of air resistance. The 60° trajectory does indeed give air drag more time and distance to act. But for the 30° trajectory the projectile spends a larger fraction of its time at speeds near the maximum speed. This suggests that the more strongly we make the resistive force increase with velocity, the more it will act to shorten the 30° trajectory relative to the 60° trajectory.

Figure 1 shows the results of a numerical computation to test this idea. The figure shows 30° and 60° trajectories for two types of resistance: (i) "standard" air drag proportional to v^2 , where v is the speed of the projectile; and (ii) drag

proportional to v^8 . In both cases, the resistive force acts antiparallel to the velocity vector. The results show—as predicted by the above argument—that for drag $\sim v^8$, the 30° trajectory is shortened more drastically by resistance than the 60° trajectory; for drag $\sim v^2$, the 60° trajectory is more strongly affected.

The above results encourage further investigation into the way trajectories are affected. To do this, we have computed trajectories based on the following equation of motion for drag forces proportional to the *n*th power of projectile speed:

$$\frac{d\mathbf{v}}{dt} = -\frac{\lambda}{m} \,\mathbf{v} v^{n-1} - g \,\hat{\mathbf{j}},\tag{1}$$

where **v** is the velocity of a particle of mass m, which is affected by a gravitational acceleration g, taken to act in the negative y direction. We limit our attention to drag forces that vary as a power of the projectile speed.

It is useful to note at the outset some scaling simplifications. If we denote the initial speed of the projectile by v_0 , then all length scales can be scaled by the natural length scale v_0^2/g of the problem. For the dynamics, we scale time by the natural time scale v_0/g , replacing t by $T \equiv tg/v_0$. Thus, if we introduce the dimensionless velocity $\mathbf{u} \equiv \mathbf{v}/v_0$, Eq. (1) reduces to

$$\frac{d\mathbf{u}}{dT} = -k\mathbf{u}u^{n-1} - \hat{\mathbf{j}},\tag{2}$$

where $k \equiv \lambda v_0^n / mg$ is the ratio of the initial drag force to the weight of the projectile. Since the initial condition in Eq. (2) is $u_0 = 1$, the problem is completely specified by the drag parameters *k* and *n*, and by the launch angle θ .

Numerical solution of the above equation of motion confirms that aiming high is often the way to go far. In addition to debunking the "obvious" nature of the "always aim low" philosophy, the numerical results, and other considerations, lead to the following observations about optimal angles.

(i) The optimal angle of launch is *greater than* 45° when n, the exponent of the velocity dependence, is larger than some critical value n_{crit} , which is around 3.5.



Fig. 1. Trajectories are shown for 30° and 60° launch angles, in the case of two types of drag; one for which n=2, and one for which n=8. The n=2 drag has a stronger shortening effect on the 60° trajectory, whereas the n=8 drag has a stronger shortening effect on the 30° trajectory. The x,y coordinates are normalized by the natural length scale v_0^2/g of the problem, where g is the acceleration of gravity and v_0 is the initial speed of the projectile. For both the n=2 and n=8 drag laws, the dimensionless indicator of damping strength, $\lambda v_0^n/mg$, has been taken to be unity. Note also that x=1 corresponds to maximum range for the drag-free case.

The precise value of n_{crit} depends on the strength of the drag (i.e., on k), increasing as drag strength increases.

- (ii) For *n* much less than n_{crit} and strong drag, the trajectories are very skewed; the descent from the maximum height is much steeper than the ascent to that height. There is no such distortion of the trajectories when *n* is much greater than n_{crit} . This point is illustrated in Fig. 2.
- (iii) For n = 8 and k = 100, the optimal angle is found to be 47.0°. For $n > n_{crit}$ parameters could not be found for which the optimal angle is much larger than 47°. This is quite different from the $n < n_{crit}$ case in which strong drag leads to very shallow optimal angles.
- (iv) For *extremely* strong drag and large n, the optimal angle is, as in the case of small n, less than 45° .

Most of these results can be understood as more than just computer output. The most basic issue, the n dependence of the optimal angle, can be understood with a calculation based on the two ways in which drag affects the range: First, it reduces the time the projectile is in the air, and second, it reduces the horizontal velocity. This can be quantified with a calculation in the limit of weak drag, i.e., a calculation to first order in the drag parameter k. This weak-limit calculation is useful in that it gives a definitive proof that θ_{opt} can be more than 45°, a proof that is independent of the difficulties that can cloud numerical results. This calculation unfortunately is not light reading, and has been relegated to the Appendix. It is recommended only to readers with the requisite skepticism and tolerance for details. The results of that calculation are rather more interesting than the calculation itself, and are shown in Fig. 3. Since the deviation of the optimal angle from 45° is proportional to k, we plot $\delta\theta/k$



Fig. 2. The shape of 30° trajectories for various drag laws. The low *n* laws produce a very skewed orbit with a descent much steeper than ascent. For large *n*, the trajectories are very nearly parabolic. The *x*, *y* coordinates are normalized by the natural length scale v_0^2/g of the problem, where *g* is the acceleration of gravity and v_0 is the initial speed of the projectile.

 $\equiv (\theta_{opt} - 45^\circ)/k$, where θ_{opt} is the angle for maximum range. A key result is that $n_{crit} = 3.4148...$ in the limit of weak drag.

The calculation in the Appendix, then, can be taken as an "explanation" of observation (i) above. Explanations of observations (ii)–(iv) above lie in a rather simple picture of the effect of air drag for large n. In the case of large n, the drag—if it is of any importance at all—is ferociously strong at the beginning of the launch, immediately slows down the projectile to a speed at which the drag is a small force compared to the weight of the projectile, and thereafter is unimportant. Large n drag, therefore, is confined to a very small portion of the beginning of the trajectory. This by itself gives an immediate explanation of observation (ii) and of the results presented in Fig. 2. A calculation based on this picture explains the remaining observations.

To do the calculation, let us imagine that the drag is effective only during the very small initial portion of the trajectory pictured in Fig. 4, and that the subsequent motion is a drag-free parabola. Let us denote by v_{trans} the speed of the projectile at which the drag/no-drag transition occurs, and let us denote by H the height at which the transition occurs. Further, let us suppose that during the strong drag phase the velocity vector is rotated downward by gravity an amount $\delta \theta_{\text{grav}}$. If a projectile is launched from height H, the angle for maximum range to a target at zero height is less than 45°. It is straightforward to show that for small H (i.e., for $H \ll v_{\text{trans}}^2/g$) the optimal angle is $45^\circ - \delta \theta_H$, where

$$\delta\theta_H = \frac{1}{2}gH/v_{\text{trans}}^2.$$
(3)



Fig. 3. The deviation of the optimal angle from 45° as a function of the drag exponent *n*, in the limit of weak drag. The deviation $\delta\theta$ is proportional to *k*, the dimensionless constant expressing the ratio of the initial drag force to the weight of the projectile. Note that the optimal angle changes from below 45° to above 45° at around n = 3.41.

If the time for the projectile to ascend to the transition is τ , then, due to the influence of gravity, the projectile will have a velocity vector at the transition that is rotated downward from its original direction by an amount

$$\delta\theta_{\rm grav} = g \,\tau / \sqrt{2} v_{\rm trans} \,, \tag{4}$$

where we have assumed that the angle during the drag phase was approximately 45° .

The important question is whether $\delta\theta_H$ or $\delta\theta_{\text{grav}}$ is larger. If $\delta\theta_H$ is larger, then one needs to adjust the initial firing angle to somewhat below 45°, so that the projectile will be optimally aimed when it starts its drag-free motion. If, on the other hand, $\delta\theta_{\text{grav}}$ is larger, gravity rotates the velocity vector



Fig. 4. The transition from drag-dominated to drag-free motion in the case of very large n.

too far down, and one must fire a bit above 45° for the projectile to be optimally aimed when it starts its drag-free motion.

To get insight into the relative size of $\delta\theta_H$ and $\delta\theta_{\text{grav}}$, let us write $H = \overline{a}\tau^2/2\sqrt{2}$, where \overline{a} is the appropriate average acceleration during the drag phase. In terms of \overline{a} , we have

$$\frac{\delta\theta_H}{\delta\theta_{\rm erav}} = \frac{\bar{a}\tau}{4v_{\rm trans}}.$$
(5)

Roughly speaking, $\overline{a\tau}$ is the amount by which the speed of the projectile was reduced during the drag phase. For weak drag, this reduction will be a small fraction of the initial speed v_0 , and we will have $\overline{a\tau} \ll v_0 \approx v_{\text{trans}}$. This argument then predicts that for weak drag and large *n*, the ratio in Eq. (5) will be much smaller than unity, and hence one must aim higher than 45° for maximum range. This conclusion is in agreement with the weak drag analysis of the Appendix.

On the other hand, for very strong drag-i.e., drag for which $v_{\text{trans}} \ll v_0$ —the ratio in Eq. (5) will be large. In this case, the need for a reduction in the angle due to the height is larger than the rotation of the velocity by gravity. This argument, then, predicts that in the case of large n and very strong drag, the optimal angle is less than 45°, just as it is for small n drag of any strength. This prediction has been confirmed with numerical integration of the equations of motion. The numerical problem is delicate since very large *n* requires a very small step size in time. To find numerical solutions, we were forced to use an adaptive step-size routine and a very small value of large n. More specifically, we used n= 4, which is a value of n just large enough so that for weak drag the optimal angle is above 45°. We computed trajectories for k = 0.1 and for $k = 10^{15}$, and found an optimal angle above 45° for the first case, and approximately 42° for the second.

This strong drag reversal explains why, for large n, it is impossible to have an optimal angle much above 45°. For small k (i.e., for weak drag) the increase of the optimal angle above 45° is proportional to k. If we try to increase the optimal angle by increasing k, we leave the weak drag regime and find that we are in fact decreasing the optimal angle. It almost seems that there is a moral lesson here about trying to aim too high, but a consideration of that hypothesis goes beyond the scope of this paper.

ACKNOWLEDGMENTS

We thank Apostol Gramada for suggesting that the optimal angle might depend on the details of drag, and George Williams for discussions and for useful suggestions. We also thank Keith Lockitch for supplying us with the adaptive stepsize code, which we used to verify the large n, strong drag prediction. We gratefully acknowledge the support of the National Science Foundation under Grant Nos. PHY-9507719 at Utah and PHY-9507742 at Wisconsin.

APPENDIX

Here we analyze the solutions of Eq. (2) to first order in k for a projectile starting with velocity components $(u_{x0}, u_{y0}) = (\cos \theta, \sin \theta)$. We start with the y (vertical) component of the equation, initially limited to the time T=0 to $T=T_1$, during which the projectile is rising:

$$\frac{du_y}{dT} = -1 - ku_y [u_x^2 + u_y^2]^p.$$
 (6)

Here we have introduced $p \equiv (n-1)/2$. Actually, for analyzing the ascending phase of the motion, it is more convenient to work with the time-reversed equation:

$$\frac{d\widetilde{u}_{y}}{d\widetilde{T}} = -1 + k\widetilde{u}_{y} [\widetilde{u}_{x}^{2} + \widetilde{u}_{y}^{2}]^{p},$$
⁽⁷⁾

where $\tilde{u_x} \equiv dx/d\tilde{T}$ and $\tilde{u_y} \equiv dy/d\tilde{T}$. Equation (7) is obtained from Eq. (6) by introducing a new time variable $\tilde{T} \equiv T_1$ -T. [A tilde ($\tilde{}$) is used to distinguish a time-reversed variable from its "forward-in-time" counterpart.] In this description, the projectile descends from time $\tilde{T}=0$ to \tilde{T} $=T_1$, starting with vertical velocity $\tilde{u_y}=0$ and ending with $\tilde{u_y}=-u_{y0}$. We chose to solve the time-reversed equation because of its simpler initial condition.

We can find \tilde{u}_y as a function of \tilde{T} by approximating $\tilde{u}_x \approx -u_{x0}$ (correct to lowest order in k) and integrating Eq. (7):

$$\widetilde{T} = \int_0^{\widetilde{T}} d\overline{T} = \int_0^{\widetilde{u}_y} \frac{d\overline{u}_y}{-1 + k\overline{u}_y [u_{x0}^2 + \overline{u}_y^2]^p}.$$
(8)

To first order in k, this gives

$$\widetilde{T} = -\widetilde{u}_{y} - k \int_{0}^{\widetilde{u}_{y}} \overline{u}_{y} [u_{x0}^{2} + \overline{u}_{y}^{2}]^{p} d\overline{u}_{y}$$
⁽⁹⁾

$$= -\tilde{u}_{y} + \frac{k}{2(p+1)} \left[u_{x0}^{2(p+1)} - (u_{x0}^{2} + \tilde{u}_{y}^{2})^{(p+1)} \right]$$
(10)

$$= -\tilde{u_y} + \frac{k}{2(p+1)} \left[u_{x0}^{2(p+1)} - (u_{x0}^2 + \tilde{T}^2)^{(p+1)} \right], \quad (11)$$

where the approximation $\tilde{u_y} \approx -\tilde{T}$ (correct to lowest order in k) has been used on the right-hand side. We can now find the maximum height H of the trajectory by integrating $\tilde{u_y}$ from $\tilde{T} = T_1$ (when the projectile is at ground level) to $\tilde{T} = 0$ (when the projectile is at its peak):

$$H = \int_{T_1}^{0} \tilde{u}_y d\tilde{T}$$
(12)
$$= \frac{1}{2} T_1^2 - \frac{k}{2(p+1)} \left[u_{x0}^{2(p+1)} T_1 - \int_0^{T_1 \approx u_{y0}} (u_{x0}^2 + \tilde{T}^2)^{p+1} d\tilde{T} \right]$$
(13)

$$= \frac{1}{2} T_1^2 - \frac{k}{2(p+1)} \left[u_{x0}^{2(p+1)} T_1 - u_{y0}^{2p+3} \times \int_0^1 (\cot^2\theta + \xi^2)^{p+1} d\xi \right].$$
(14)

Here, we have replaced u_{x0}/u_{y0} by $\cot \theta$, where θ is the angle at which the projectile is launched.

We now consider the descent of the projectile, and reset the (forward-in-time) clock so that it starts at T=0. We let T_2 stand for the time at which the projectile reaches ground level (i.e., the starting height) again. For this case, we choose to solve the forward-in-time Eq. (6). This equation is identical to Eq. (7) except for the sign of the drag term. Also, the initial condition $(u_y=0 \text{ at } T=0)$ in terms of the forward-intime variables is identical to that for the ascent of the projectile in terms of the time-reversed variables. Thus, by simply reversing the sign of the drag term and replacing T_1 everywhere by T_2 , it follows immediately from Eq. (14) that *H* is also given by

$$H = \frac{1}{2} T_2^2 + \frac{k}{2(p+1)} \left[u_{x0}^{2(p+1)} T_2 - u_{y0}^{2p+3} \times \int_0^1 (\cot^2 \theta + \xi^2)^{p+1} d\xi \right].$$
 (15)

By subtracting (14) from (15), and using $\frac{1}{2}(T_2^2 - T_1^2) \approx u_{y0}(T_2 - T_1)$ (which is correct to first order in k), it follows that

$$T_2 - T_1 = \frac{k}{(p+1)} \left[u_{y0}^{2(p+1)} \int_0^1 (\cot^2 \theta + \xi^2)^{p+1} d\xi - u_{x0}^{2(p+1)} \right].$$
(16)

Also, by setting $\tilde{u_y} = -u_{y0}$ at $\tilde{T} = T_1$ in Eq. (10), we get

$$2T_1 = 2u_{y0} - \frac{k}{(p+1)} \left[1 - u_{x0}^{2(p+1)}\right].$$
(17)

Finally, we add these last two results to obtain T_{tot} , the total time of flight of the projectile:

$$T_{\text{tot}} \equiv T_1 + T_2 = 2u_{y0} + \frac{k}{(p+1)} \times \left[u_{y0}^{2(p+1)} \int_0^1 (\cot^2 \theta + \xi^2)^{p+1} d\xi - 1 \right].$$
(18)

Next we consider the equation

$$\frac{du_x}{dT} = -ku_x [u_x^2 + u_y^2]^p \tag{19}$$

for the horizontal motion, and in the drag term make the lowest order approximation $u_x \approx u_{x0}$ and $u_y \approx (u_{y0} - T)$:

$$\frac{du_x}{dT} = -ku_{x0}[u_{x0}^2 + (u_{y0} - T)^2]^p.$$
(20)

We now integrate this equation to arrive at

$$u_{x} = u_{x0} - ku_{x0} \int_{0}^{T} [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} d\overline{T}.$$
 (21)

To find the range *R*, we integrate u_x from T=0 to $T=T_{tot}$, using the approximation $T_{tot} \approx 2u_{y0}$ in terms that are already first order in *k*:

$$R = u_{x0}T_{tot} - ku_{x0}\int_{0}^{2u_{y0}} dT \int_{0}^{T} [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} d\overline{T}.$$
(22)

The double integral in the expression can be simplified by changing the order of integration:

$$\int_{0}^{2u_{y0}} dT \int_{0}^{T} [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} d\overline{T}$$
$$= \int_{0}^{2u_{y0}} d\overline{T} \int_{\overline{T}}^{2u_{y0}} [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} dT$$
(23)

$$= \int_{0}^{2u_{y0}} (2u_{y0} - \overline{T}) [u_{x0}^{2} + (u_{y0} - \overline{t})^{2}]^{p} d\overline{T}$$
(24)

$$= \int_{0}^{2u_{y0}} u_{y0} [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} d\overline{T} + \int_{0}^{2u_{y0}} (u_{y0} - \overline{T}) \\ \times [u_{x0}^{2} + (u_{y0} - \overline{T})^{2}]^{p} d\overline{T}.$$
(25)

The second integral above vanishes by symmetry, and the first can be written as

$$2u_{y0}^{2(p+1)} \int_0^1 (\cot^2 \theta + \xi^2)^p d\xi.$$
 (26)

We can now use this result, and Eq. (18), in Eq. (22) to arrive at an expression for the range that is correct to first order in k. We express the result in terms of the launch angle θ , replacing u_{x0} , u_{y0} by $\cos \theta$, $\sin \theta$. The result is

$$R = 2\sin\theta\cos\theta + k\mathscr{F}(\theta), \tag{27}$$

where

$$\mathscr{F}(\theta) = \frac{\cos \theta}{(p+1)} \left[(\sin \theta)^{2(p+1)} \int_0^1 (\cot^2 \theta + \xi^2)^{p+1} d\xi - 1 \right] - 2 \cos \theta (\sin \theta)^{2(p+1)} \int_0^1 (\cot^2 \theta + \xi^2)^p d\xi.$$
(28)

To find the critical value of *n*, we now differentiate $\mathscr{F}(\theta)$ with respect to θ and evaluate it at 45°:

$$\mathscr{F}'|_{45^{\circ}} = \frac{1}{\sqrt{2}} \left[\frac{1}{(p+1)} \left(1 + \frac{2p+1}{2^{p+1}} \int_{0}^{1} (1+\xi^{2})^{p+1} d\xi \right) - \frac{2p+3}{2^{p}} \int_{0}^{1} (1+\xi^{2})^{p} d\xi + \frac{2p}{2^{(p-1)}} \int_{0}^{1} (1+\xi^{2})^{p-1} d\xi \right].$$

$$(29)$$

This expression is easily evaluated numerically and is found to be negative for small p, changing to positive at p = 1.2074.... This means that in the limit of weak drag, the optimal angle changes from less than 45° to greater at n = 2p + 1 = 3.4148....

For weak drag the optimal angle can be written as $45^{\circ} + \delta\theta$, where $\delta\theta$ is first order in *k*. From the definition of \mathscr{F} it follows that $\delta\theta/k = \frac{1}{4}\mathscr{F}'|_{45^{\circ}}$. It is this quantity that is plotted in Fig. 3. As a check, we computed $\delta\theta/k$ directly without simplifying the double integrals occurring in Eq. (22). The results were in excellent agreement with those given by Eq. (29) and in Fig. 3.

¹We do not wish to wander too deeply into real world effects. For projectiles traveling very large distances, the decrease in atmospheric density favors high angles of launch. [See D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics* (Wiley, New York, 1981), 5th ed., Question 16, p. 73.] Here we consider a flat earth and homogeneous atmosphere.

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⁵C. W. Groetsch, "On the optimal angle of projection in general media," Am. J. Phys. **65** (8), 797–799 (1997).

⁶Richard H. Price and Joseph D. Romano, "Comment on 'On the optimal angle of projection in general media," by C. W. Groetsch," Am. J. Phys. 66 (2), 114 (1998.

SCORNING THE BASE DEGREES

We need notice at the moment only that the choice of the simplest law that fits the facts is an essential part of procedure in applied mathematics, and cannot be justified by the methods of deductive logic. It is, however, rarely stated, and when it is stated it is usually in a manner suggesting that it is something to be ashamed of. We may recall the words to Brutus.

But 'tis a common proof That lowliness is young ambition's ladder, Whereto the climber upward turns his face; But when he once attains the upmost round, He then unto the ladder turns his back, Looks in the clouds, scorning the base degrees By which he did ascend.

Harold Jeffreys, Theory of Probability (Oxford University Press, 1939), p. 4.