

## Sec. 14.5: Chain Rule

### ① Explicit differentiation

Recall from Calc. I:

If  $z = f(x)$  and  $x = x(t)$ , then

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \quad \leftarrow \begin{array}{l} \text{Calc. I} \\ \text{Chain Rule} \end{array}$$

Our goal in this topic will be to **extend** this 1-variable rule (i.e., each of  $f$  and  $x$  depend only one 1 variable) **to the multi-variable case**.

Case 1(a):

$$z = f(x(t), y(t))$$

Interpretation:

Car on the road;

Road is on the hill  $z = f(x, y)$ ;

In  $(x, y)$ -plane, the car's position is given by  $(x(t), y(t))$ .

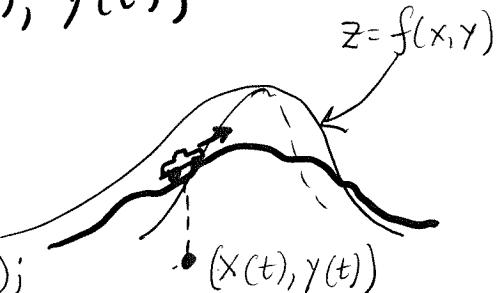
One may then ask: How does the elevation (i.e.  $z$ -coordinate) of the car changes with  $t$ ?

Formula:

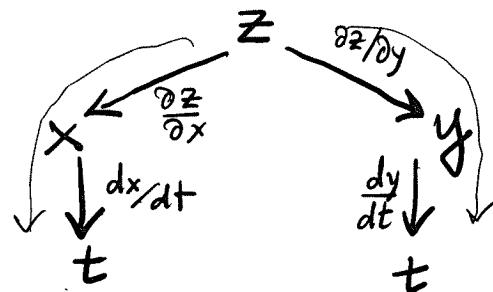
$$\frac{dz}{dt} \equiv \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

"same as"

See Ex. 1/Book  
for numbers.



Tree diagram:



Comments:

1) Always write the Tree Diagram before writing out the formula.

In the Tree Diagram, trace all paths that lead from  $z$  to  $t$  (2 such paths in the case above).

2) Note:  $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$  are partial derivatives,

but  $(\frac{dx}{dt}, \frac{dy}{dt})$  are ordinary derivatives. Why?

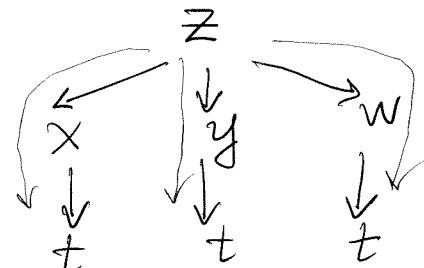
Because  $f(x, y)$  depends on more than one variable,  $\Rightarrow$  need partials.

On the other hand, each of  $x(t), y(t)$  depends on one variable,  $\Rightarrow$  can only have ordinary derivative.

3) If  $f$  depends on more than 2 variables, the formula is similar:

$$\frac{df(x(t), y(t), w(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dt}$$

etc.

Tree Diagram:

(3 paths from  $z$  to  $t$ )

Case 1(b): In the last formula on the previous page,

let's set  $w(t) = t$ . Then we have:

$$\frac{df(x(t), y(t), t)}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial w} \cdot \boxed{\frac{dw}{dt} \stackrel{t}{\rightarrow}}$$

but:  $w=t$        $\frac{dt}{dt} = 1$

Thus:

$$\boxed{\frac{df(x(t), y(t), t)}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial t}}$$

Ex. 1  $f(x, y, t) = t x^2 + \sin(xy)$ ,  $x=3\sqrt[3]{t}$ ,  $y=e^{5t}$ .

Find  $df/dt$ .

Sol'n: Use the formula of Case 1(b):

$$\begin{aligned} \frac{df}{dt} &= \underbrace{\left( t \cdot 2x + y \cdot \cos(xy) \right)}_{\frac{\partial f}{\partial x}} \cdot \underbrace{\frac{3}{2\sqrt[3]{t}}}_{\frac{dx}{dt}} + \\ &+ \underbrace{x \cdot \cos(xy)}_{\frac{\partial f}{\partial y}} \cdot \underbrace{5e^{5t}}_{\frac{dy}{dt}} + \underbrace{x^2}_{\frac{\partial f}{\partial t}}. \end{aligned}$$

(Can simplify, but this won't be required on quiz/test.)

Case 2  $z=f(x, y)$ ,  $x=x(u, v)$   
 $y=y(u, v)$

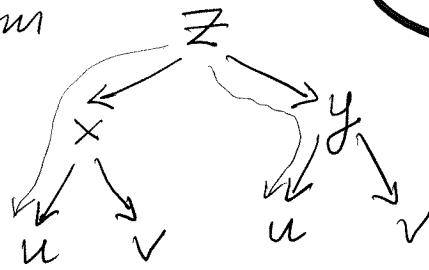
To obtain the formula for, say,  $\frac{\partial z}{\partial u}$ , draw the Tree Diagram.

There are 2 paths leading from  $z$  to  $u$ . So:

$$\frac{\partial f(x(u,v), y(u,v))}{\partial u}$$

=

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$



and similarly for  $\frac{\partial f}{\partial v}$ .

Note: All derivatives here are partial (not ordinary) because all variables:  $f, x, y$  depend on more than 1 variable (see the Tree Diagram).

Ex. 2 (similar to Ex. 3 in book):

$$\text{Let } z = x^5 e^{y^2}, \quad x = r \cos \theta, \quad y = r \sin \theta.$$

$$\text{Find } \frac{\partial z}{\partial \theta}.$$

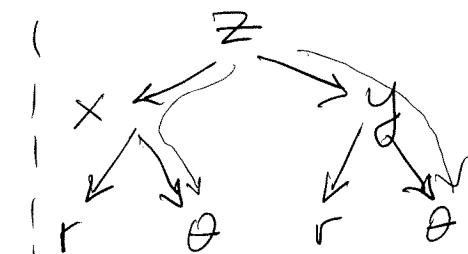
Sol'n:

$$\frac{\partial z}{\partial \theta} \stackrel{\uparrow}{=} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

from Tree

$$\begin{aligned} &= (5x^4 \cdot e^{y^2}) \cdot (-r \sin \theta) + \\ &\quad (x^5 \cdot 2y \cdot e^{y^2}) \cdot (r \cos \theta). \end{aligned}$$

Tree:



One can, of course, simplify this expression, but this will not be required on test/quiz.

Case 2 for functions of 3 or more variables is handled similarly; See Ex. 4, 5 / book.

## ② Implicit differentiation.

Review of Calc. I situation (recall that you were required to read Ex. 5 in Sec. 14.3 to review this material):

Find the slope of circle  $x^2 + y^2 = R^2$  at a given point  $(x, y)$  on the circle.

Sol'n: On the circle, we have  $y = y(x)$ .

Differentiate the equation of circle w.r.t.  $x$ :

$$\frac{d}{dx} (x^2 + y(x))^2 = R^2 \Rightarrow$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

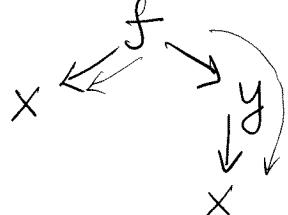
See also Ex. 8 in textbook (Sec. 14.5).

We can generalize this from a circle to any curve:

$f(x, y) = 0$  defines some curve  $y = y(x)$   
 $(x^2 + y^2 - R^2 = 0$  in the above case of the circle).

$$\frac{d}{dx} (f(x, y(x)) = 0) \Rightarrow \frac{\partial f}{\partial x} \cdot \cancel{\frac{dx}{dx}} + \frac{\partial f}{\partial y} \cdot \cancel{\frac{dy}{dx}} = 0$$

↑  
Case 1 in topic 1



$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

And now we can generalize to 3 variables:

$F(x, y, z) = 0$  defines some surface  $z = z(x, y)$ .

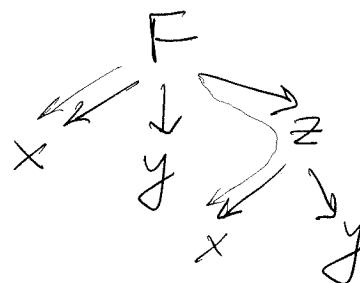
(think of sphere  $x^2 + y^2 + z^2 - R^2 = 0$ )

To find the slope of this surface along, say,  $x$ -axis:

$$\frac{\partial}{\partial x}(F(x, y, z(x, y)) = 0) \Rightarrow$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$



See posted page 13-4 and Ex. 9 in book.

### ③ Application to rates of change

MUST READ Ex. 2 in textbook.

### ④ Application to partial differential equations

Ex. 3 Show that for any function  $f(u)$ , the function  $z = f(x + 2y)$  satisfies a partial diff. equation  $2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ .

Interpretation (vague): This P.D. equation is a special case of the Wave Eqn. (sec. 14.3). The problem "shows" (vague!) that a disturbance of any shape can propagate along a tight string (vague!!).

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Sol'n:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \xrightarrow{f(u)} = \frac{df}{du} \cdot \frac{\partial u}{\partial x} \xrightarrow{(x+2y)} \\ &= \frac{df}{du} \cdot 1 \\ \frac{\partial z}{\partial y} &= \frac{df}{du} \cdot \frac{\partial u}{\partial y} \xrightarrow{(x+2y)} = \frac{df}{du} \cdot 2. \\\text{similarly}\end{aligned}$$

$$\begin{array}{c} z \\ \downarrow \\ u = x + 2y \\ \swarrow \quad \searrow \\ x \quad y \end{array}$$

Thus,

$$2 \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2 \cdot \frac{df}{du} \cdot 1 - \frac{df}{du} \cdot 2 = 0. \quad \checkmark$$

See also Ex. 6 / book (it is more complicated).