

Sec. 16.5. Curl and divergence.

① Definitions.

Recall the gradient: $\vec{\nabla} f(x, y, z) = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z}$.

Define vector $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Name: $\vec{\nabla}$ = nabla (not Greek)

It may look formal but:

- Has all properties of a vector hence can be treated as such (power of abstraction!).
- In higher-level (grad) courses it is shown that in some way, $\vec{\nabla}$ behaves just like a regular vector $\langle a, b, c \rangle$.

$\vec{\nabla} f$ shows how vector $\vec{\nabla}$ acts on a scalar function f .

How does $\vec{\nabla}$ act on a vector function \vec{F} ?

Curl:

$$\vec{\nabla} \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

(see Ex. 1 in book for numbers).

Divergence:

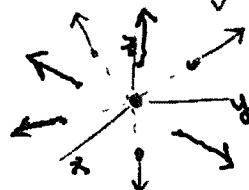
$$\vec{\nabla} \cdot \vec{F} = \text{div } \vec{F} = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R.$$

(see Ex. 4 in book).

② Meaning of curl & div.

Ex. 1 Find curl & div of $\vec{F} = \vec{r} = \langle x, y, z \rangle$

o) Draw:



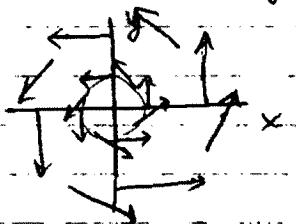
This vector field diverges from $(0, 0, 0)$. (Similarly to the 2D Ex. 1/Notes for Sec. 16.1.)

$$1) \operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(yz^0 - yz^0) - \vec{j}(xz^0 - xz^0) + \vec{k}(xy^0 - xy^0) = \vec{0}$$

$$2) \operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3.$$

Ex. 2 Find curl & div of $\vec{F} = \langle -y, x, 0 \rangle$

0) Sketch:
(Ex. 2 in Sec. 16.1)
(Notes)



This field swirls around origin.

$$1) \operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \vec{i}(0-x_z) - \vec{j}(0+y_z) + \vec{k}(x_x + y_y) = 2\vec{k}.$$

$\operatorname{curl} \vec{F} = 2\vec{k}$ — axis of rotation

$$2) \operatorname{div} \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}0 = 0$$

The curl is directed along the axis of rotation and its sign follows the right-hand rule (see Sec. 12.4.).

Moral: • $\operatorname{div} \vec{F}$ measures the tendency of \vec{F} to diverge from its source (or to converge into a sink).

E.g., an expanding gaseous fireball diverges from its center.

- The ability to diverge (converge) is related to the ability to expand (or to contract). Therefore, if with $\operatorname{div} \vec{F} = 0$, \vec{F} describes an incompressible motion (e.g., of a fluid).

- A purely diverging \vec{F} is irrotational, i.e. $\text{curl } \vec{F} = \vec{0}$. (Ex. 1).
- $\text{curl } \vec{F}$ measures the tendency of \vec{F} to rotate. (Irrotational field has $\text{curl } \vec{F} = \vec{0}$.)
- A purely rotational field has zero divergence ($\text{div } \vec{F} = 0$); Ex. 2.

③ Compositions of two \vec{F} 's.

(a) Let $\vec{F} = \text{curl } \vec{G} = \vec{\nabla} \times \vec{G}$ for some \vec{G} .

$$\text{Then } \text{div } \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{G}) = 0$$

Sec 12.4 \rightarrow "1" $\vec{\nabla}$ Sec. 12.3

(verified by direct calculation in the book)

Conclusion: A given \vec{F} can be represented as $\vec{F} = \text{curl } \vec{G}$ for some \vec{G} if and only if $\text{div } \vec{F} = 0$.

The proof is above.
See Ex. 5 in book for #. (Also, agrees with the last bullet point of "Moral" above.)

(b) Let $\vec{F} = \vec{\nabla} f$ for some f .

Then

$$\begin{aligned} \text{div } \vec{F} &= \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z \\ &= f_{xx} + f_{yy} + f_{zz} = \nabla^2 f \end{aligned}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called Laplacian and plays a major role in all branches of physics and engineering.

(c) Again, let $\vec{F} = \vec{\nabla} f$.

Then

Sec. 16.1!
 \vec{F} is a conservative field.

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{\nabla} f = \vec{0}$$

Sec. 12.4

(verified by direct calculation in the book)

Conclusion:A given \vec{F} can be represented as $\vec{F} = \vec{\nabla} f$ for some f if and only if $\text{curl } \vec{F} = \vec{0}$.

See Ex. 2, 3(a) in book.

Will study this again in Sec. 16.3 & 16.4.

(4) Elementary properties

$$\text{div } (\vec{F} + \vec{G}) = \text{div } \vec{F} + \text{div } \vec{G}$$

$$\text{curl } (\vec{F} + \vec{G}) = \text{curl } \vec{F} + \text{curl } \vec{G}$$

Other properties - see #25, 26, 27 after Sec. 16.5; they are proved using the product rule.

(5) Note about some incorrect notations.

Recall that $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$
 $= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. Note the order of the derivative and the function in each term: e.g., in $\frac{\partial}{\partial x} P$: the derivative is written before the function! (this is how we have always written derivatives in this course when we used the $\frac{\partial}{\partial x}$ etc. notations.) However, for an unknown reason, I have seen many students write:

$$\text{div } \vec{F} = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} \quad \leftarrow \text{THIS IS WRONG!}$$

The notation $\frac{\partial P}{\partial x}$ has a clear meaning: $\frac{\partial}{\partial x}$ is taken of P .On the other hand, the notation $P \frac{\partial}{\partial x}$ means something totally different, that we did not study in this course!So, do not use this wrong notation in either $\text{div } \vec{F}$ or $\text{curl } \vec{F}$!