

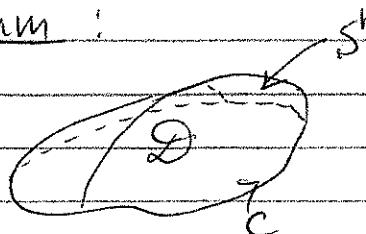
Sec. 16.9. Divergence Thm.
 (Gauss Thm.)

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① Main result.

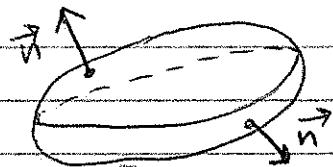
It extends Green's / Stokes Thm. to 3D.

Stokes Thm:



$$\oint \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

Divergence Thm:



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$$

$E = 3D$ solid

$S =$ its boundary
(closed surface)

Flux through S ,

Thm: Let E be a simply connected solid and S be the boundary (surface) of E , with positive orientation (outside).

Let \vec{F} have continuous partial derivatives in an open region that contains E .

Then:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \cdot dV$$

Flux through boundary of E .

Note: This is an extension of FTC to 3D; see Sec. 16.4;
 also the Summary (Sec 16.10)
 in textbook

Proof

0) Rewrite the statement.

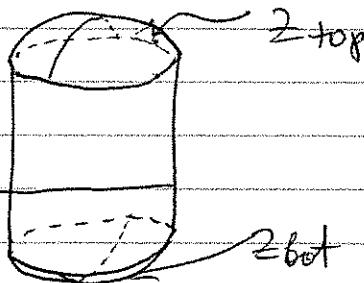
$$\vec{F} = \langle P, Q, R \rangle; \quad d\vec{s} = \vec{n} dS.$$

Then

$$\iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \iint_S (\vec{P} \cdot \vec{i} + \vec{Q} \cdot \vec{j} + \vec{R} \cdot \vec{k}) \cdot \vec{n} dS.$$

Let's move the R -part; the rest are similar.

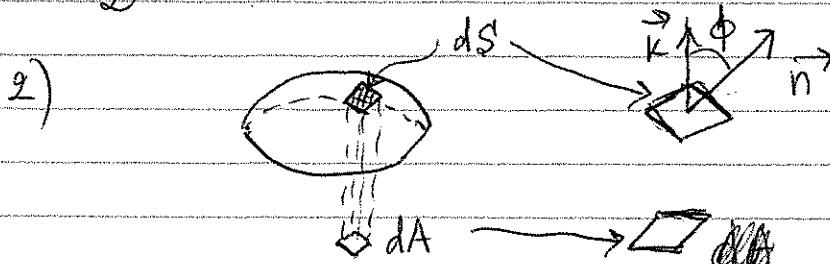
1) Suppose E is bounded by top, bottom, and vertical "walls"



$$\iiint_D \frac{\partial R}{\partial z} dV = \iint_D \left(\int_{z_{\text{bot}}}^{z_{\text{top}}} \frac{\partial R}{\partial z} dz \right) dx dy$$

projection of E on xy -plane

$$= \iint_D (R(x, y, z_{\text{top}}) - R(x, y, z_{\text{bot}})) dx dy.$$



$$dA = dS \cdot \cos \phi$$

(see Sec. 16.7)

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Then at the top surface:

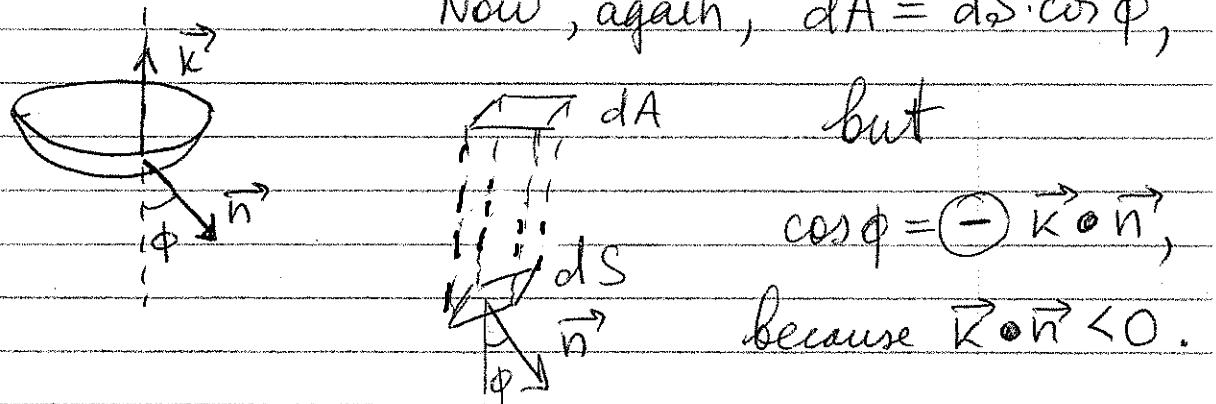
$$\iint_D R(x, y, z_{\text{top}}) dA = \iint_D R(x, y, z_{\text{top}}) \cdot \cos \phi \cdot dS$$

$$= \iint_{S_{\text{top}}} R(x, y, z) \vec{k} \cdot \vec{n} dS.$$

\uparrow
on $S_{\text{top}}, z = z_{\text{top}}$.

3) At the bottom surface:

Now, again, $dA = dS \cdot \cos \phi$,



$$\text{So } \iint_D R(x, y, z_{\text{bot}}) dA = \iint_{S_{\text{bot}}} R(x, y, z) (-\vec{k} \cdot \vec{n}) dS$$

q) Thus,

$$\iiint_E \frac{\partial R}{\partial z} dV = \iint_D (R(x, y, z_{\text{top}}) - R(x, y, z_{\text{bot}})) dA$$

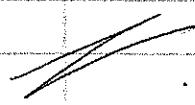
$$= \iint_{S_{\text{top}}} R \vec{k} \cdot \vec{n} dS - \iint_{S_{\text{bot}}} R \cdot (-\vec{k} \cdot \vec{n}) dS$$

$$= \iint_{(S_{\text{top}} + S_{\text{bot}})} R \cdot \vec{k} \cdot \vec{n} dS \quad (\text{on the side walls, } \vec{n} \perp \vec{k} \Rightarrow \vec{k} \cdot \vec{n} = 0)$$

\Rightarrow

$$\iiint_E \frac{\partial R}{\partial z} dV = \iint_S R \vec{F} \cdot \vec{n} dS.$$

$S =$ full boundary of E



Significance of Div. theorem:

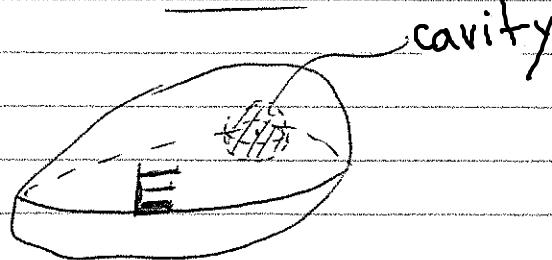
Can compute the flux of \vec{F} through closed surface S by evaluating the

$$\iiint_E \operatorname{div} \vec{F} dV, \text{ where } E \text{ is bounded by } S.$$

(often, $\operatorname{div} \vec{F}$ has a simpler form than $\vec{F} \cdot \vec{n}$.)

See Ex. 1, 2 in book.

- ② Extension of Div. Thm. when E has holes.



Similarly to a derivation in Sec. 16.4 where D had a hole:



here we have:

$$\iiint_E \operatorname{div} \vec{F} dV = (\iint_{S_{\text{outside}}} + \iint_{S_{\text{cavity}}}) \vec{F} \cdot \vec{n} dS,$$

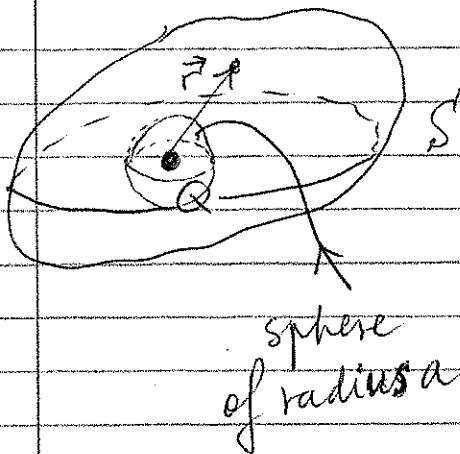
S_{outside} S_{cavity}

where \vec{n} points outside on S_{outside} ;
 \vec{n} points inside on S_{cavity} .

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Ex. 1

In electrostatics,
the following problem is
important.



Find the flux of electric field \vec{E} through any surface S enclosing a point charge Q .

Sol'n: 1) $\vec{E} = \frac{Q \vec{r}}{|\vec{r}|^3}, \vec{r} = \langle x, y, z \rangle$.

2) \vec{E} (and $\text{div } \vec{E}$) are not continuous at $\vec{r} = 0$, \Rightarrow cannot apply the Div. Thm. to the region enclosed by S .

But can apply it to a region with a hole around $\vec{r} = 0$!

So: $\iint_S (\iint_{\text{cavity}} \vec{E} \cdot d\vec{S}) = \iiint_V \text{div } \vec{E} dV$

\vec{E} with a cavity

3) Verify by direct calculation (tedious!)
that

$$\text{div } \vec{E} = Q \text{div} \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}^3} = 0 \quad (\text{for } \vec{r} \neq 0)$$

Then $\left(\iint_S (\iint_{\text{cavity}} \vec{E} \cdot d\vec{S}) \right) = \iiint_V 0 dV = 0$.

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$$\text{So } \iint_S \vec{E} \cdot d\vec{S} = - \iint_{\text{cavity}} \vec{E} \cdot d\vec{S} = + \iint_{\text{"-S cavity."}} \vec{E} \cdot d\vec{S}$$

Surface of the cavity
with \vec{n} pointing outside.

4) On S , $|r| = a$, $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$. Then

$$\iint_{\text{"-S cavity."}} \vec{E} \cdot d\vec{S} = \iint_{\text{"-S cavity."}} \frac{Q \vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} dS$$

$$= Q \iint_{\text{"-S cavity."}} \frac{\vec{r} \cdot \vec{r}}{|\vec{r}|^4} dS = Q \iint_{\text{"-S cavity."}} \frac{1}{a^2} dS$$

$$= \frac{Q}{a^2} \iint_{\text{"-S cavity."}} dS = \frac{Q}{a^2} \cdot 4\pi a^2 = 4\pi Q.$$

surface of a sphere of radius a

Thus:

$$\iint_S \frac{Q \vec{r}}{|\vec{r}|^3} \cdot d\vec{S} = 4\pi Q$$

for any surface surrounding Q

Gauss Law