

# Review of Cross-Product (Sec. 12.4) (3-1)

## ① Definition

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note: ~~vector~~  
cross-product is  
a vector!!!

## ② Main properties of cross-product

1)  $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

2) Corollary of 1):

$$\vec{u} \times \vec{u} = \vec{0}. \text{ More generally:}$$

$$\vec{u} \times (k\vec{u}) = k \cdot (\vec{u} \times \vec{u}) = \vec{0}.$$

any scalar

But  $k\vec{u} \parallel \vec{u}$  (Sec. 12.2). Thus:

$$(\vec{u} \times \vec{v} = \vec{0}) \Leftrightarrow (\vec{u} \parallel \vec{v})$$

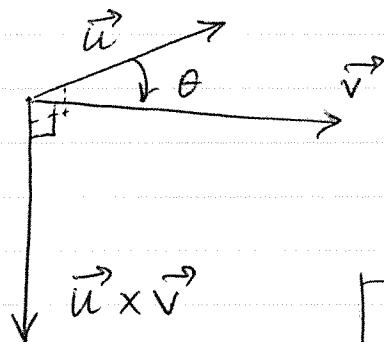
- This formula can be used to check if  $\vec{u} \parallel \vec{v}$  when we don't "see" components of  $\vec{u}$  &  $\vec{v}$  as numbers.
- (when we see them as numbers, we just do an inspection if  $\vec{v} = k \cdot \vec{u}$  for some  $k$ ).

think of this  
as  $k\vec{u}$ .

→ Other properties: **must** see p. 819 (e-book).

### ③ Geometric meaning & purpose of cross-product

3-2



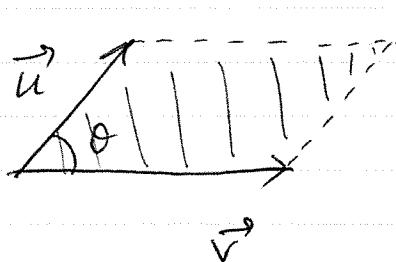
$$(\vec{u} \times \vec{v}) \perp \vec{u}$$

$$(\vec{u} \times \vec{v}) \perp \vec{v}$$

Thus,

$$(\vec{u} \times \vec{v}) \perp (\text{plane made by } \vec{u} \text{ & } \vec{v})$$

The main use of cross-product is it is a vector that is  $\perp$  to two given vectors.



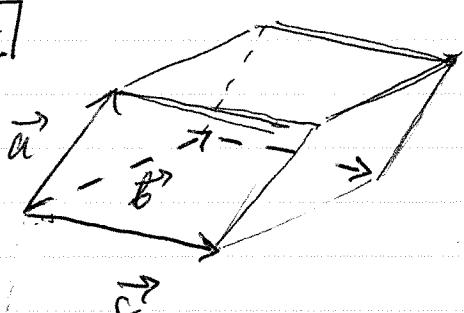
$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$$

(magnitude of  $\vec{u} \times \vec{v}$ )

(area of parallelogram made by  $\vec{u}$  &  $\vec{v}$ )

### ④ Triple product

a



$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

volume of slanted box made by  $\vec{a}, \vec{b}, \vec{c}$  in any permutation

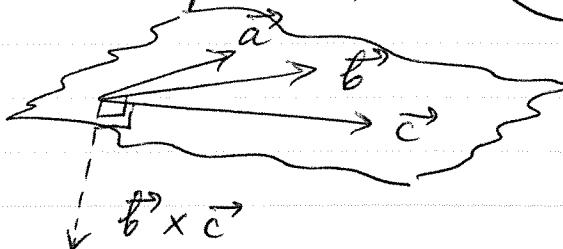
Formula:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If  $\vec{a}, \vec{b}, \vec{c}$  lie in same plane,  
then  $\nabla = 0$ :

3-3

(We can also see  
that in this case  
 $\vec{a} \perp (\vec{b} \times \vec{c})$ ,



$\Rightarrow \vec{a} \circ (\vec{b} \times \vec{c})$  must  $= 0 \leftarrow$  see Sec. 12.3.)  
thus:

$$(\vec{a}, \vec{b}, \vec{c} \text{ lie in } \begin{matrix} \\ \text{same plane} \end{matrix}) \Leftrightarrow (\vec{a} \circ (\vec{b} \times \vec{c}) = 0)$$

[See Ex. 5 in book for numbers.]

In Lecture 1, topic 2c we showed:

$$(\vec{a}, \vec{b}, \vec{c} \text{ lie in same plane } \text{ (and } \vec{a} \parallel \vec{b})) \Rightarrow (\vec{c} = s\vec{a} + t\vec{b})$$

for some  
scalars  $s, t$

Now we can show  $\Leftarrow$ .

Proof: Let  $\vec{c} = s\vec{a} + t\vec{b}$ .

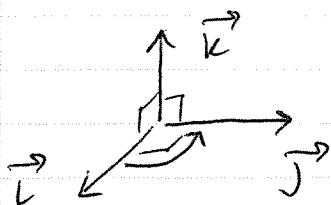
To test if  $\vec{c}$  is in same plane  
with  $\vec{a}$  &  $\vec{b}$ , compute triple prod:

$$\begin{aligned} \vec{c} \circ (\vec{a} \times \vec{b}) &= (s\vec{a} + t\vec{b}) \circ (\vec{a} \times \vec{b}) \\ &= s \cdot \underbrace{\vec{a} \circ (\vec{a} \times \vec{b})}_{\perp \vec{a}} + t \cdot \underbrace{\vec{b} \circ (\vec{a} \times \vec{b})}_{\perp \vec{b}} \\ &= s \cdot 0 + t \cdot 0 = 0. \quad \checkmark \end{aligned}$$

So,  $\vec{c}$  is in same plane as  $\vec{a}, \vec{b}$ . q.e.d.

⑤ Cross-product of  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$ .

(3-4)



$$|\vec{i} \times \vec{j}| = |\vec{i}| \cdot |\vec{j}| \cdot \sin\theta = 1.$$

$|\vec{i}| = 1, |\vec{j}| = 1, \sin 90^\circ = 1.$

Direction:  $(\vec{i} \times \vec{j}) \perp xy\text{-plane},$   
points up.  
Thus,

$\vec{i} \times \vec{j} = \vec{k}$
$\vec{j} \times \vec{k} = \vec{i}$
$\vec{k} \times \vec{i} = \vec{j}$

$$(so \quad \vec{i} \times \vec{k} = -\vec{j})$$

Also,  $\vec{i} \times \vec{i} = \vec{0} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k}$ .  
(see topic ② above).

HW #3 TF (p. 842) ④ 6, 7, 9, 13, 14, 20, 21.

Sec. 12.4: 13, 3 - meaning, def.

11 - basic properties

16 - rh.-rule

29, 39 - vector  $\perp$  to two given ones

37, 38 - coplanar