Sec. 1.7. Linear independence & Nonsingular matrices.

PART A:

① Linear independence of vectors

NOTE: The material of both parts of this section will be FUNDAMENTAL for understanding most concepts developed in the rest of the course.

Def: $\underline{\theta} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is called the zero vector.

Def: A set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ is called linearly independent

if the only sol'n to the vector eqn:

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_p \underline{v}_p = \underline{\theta} \qquad (\star)$$

is $c_1 = c_2 = \ldots = c_p = 0$.

(A set of vectors is called linearly dependent if it is not lin. independent.)

Def*: A set $\{\underline{v}_1, \ldots \underline{v}_p\}$ is lin. dependent when $(\star)$ has a sol'n where not all of $c_1, c_2, \ldots c_p$ are 0.

Q: Why the name "linearly dependent"?
(Do this for 3 vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$.)

- Intuitively, we want to say that $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are dependent when one of them depends on the other two, e.g.:

$$\underline{v}_3 = k_1 \underline{v}_1 + k_2 \underline{v}_2 \qquad (■)$$

for some k_1, k_2.

How does this jive with the Def.* above?

- Rewrite (■) as:

$$\underbrace{k_1}_{c_1} \underline{v}_1 + \underbrace{k_2}_{c_2} \cdot \underline{v}_2 + \underbrace{(-1)}_{c_3} \cdot \underline{v}_3 = \underline{\theta}$$

Of c_1, c_2, c_3, at least one ($c_3 = -1$) is not 0. ✓

So, our intuitive (not mathematical!) definition is:

$$\boxed{\begin{array}{c}\text{(Set } \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\} \text{ is lin. dependent)} \\ \Longleftrightarrow \\ \text{(one vector is a linear combination of the rest.)}\end{array}}$$

Ex 1 Determine if set $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is lin. dependent or independent:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}.$$

Sol'n: 1) Setup the definition (☆):

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta} \qquad (☆)$$

and proceed to solve for c_1, c_2, c_3.

2) From (★), use the **KEY FORMULA** of Sec. 1.5 to write (★) as:

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{\theta}$$

This is a l.s. (written in matrix form), ⇒ solve it.

3) Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & -1 & 5 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} c_1 + 2c_3 = 0 \\ c_2 - c_3 = 0 \\ c_3 = \text{free} \end{array}$$

$$\Rightarrow c_1 = -2c_3, \quad c_2 = c_3, \quad c_3 = \text{free}.$$

Since $c_3 \neq 0$ in general, we've found a sol'n of (★) with some nonzero c_1, c_2, c_3, ⇒ by the definition, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are lin. <u>dependent</u>.

- Let's check that this answer agrees with our <u>intuitive</u> definition above.

$$\underset{\substack{\uparrow \\ -2c_3}}{c_1} \underline{v}_1 + \underset{\substack{\uparrow \\ c_3}}{c_2} \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta} \quad \Rightarrow \quad \frac{-2c_3 \underline{v}_1 + c_3 \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta}.}{c_3 \ (\text{since } c_3 \neq 0)}$$

$$\Rightarrow -2\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = \underline{\theta} \quad \Rightarrow \quad \boxed{\underline{v}_3 = 2\underline{v}_1 - \underline{v}_2}. \quad \checkmark$$

<u>Ex. 2</u> The set of **unit vectors** is lin. independent:

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad \underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

See p. 75 in textbook.

Ex. 3 (This Example will be extensively used later)

For what value of a is set $\{\underline{v}_1, \underline{v}_2\}$ lin. independent, where $\underline{v}_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

Sol'n: (follow the lines of Ex. 1)

1) Setup: $c_1 \underline{v}_1 + c_2 \underline{v}_2 = \underline{\theta}$; seek c_1, c_2.

2) l.s. in matrix form: (by the Key Formula) $[\underline{v}_1, \underline{v}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underline{\theta}$

3) Sol'n using augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ a & 3 & 0 \end{array}\right) \xrightarrow[R_2 - aR_1 \to R_2]{} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3-2a & 0 \end{array}\right) \ldots$$

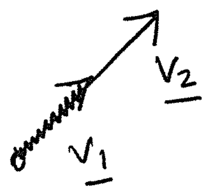
Need to consider 2 cases:

- $3-2a \neq 0$ $(a \neq 3/2)$
  $\Rightarrow$ can divide by $(3-2a)$, $\Rightarrow$ get $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{REF} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$
  $\Rightarrow$ $c_1 = 0$, $c_2 = 0$ is the <u>only</u> sol'n of (☆), $\Rightarrow$ vectors are lin. independent (for $a \neq 3/2$).
- $3-2a = 0$ $(a = 3/2)$
  $\Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} c_1 + 2c_2 = 0 \\ c_2 = \text{free} \end{array} \Rightarrow \begin{array}{l} c_1 = -2c_2 \\ c_2 = \text{free} \end{array}$,
  $\Rightarrow$ (☆) has nonzero sol'ns $(c_1 = -2c_2 \neq 0)$, $\Rightarrow$ the vectors are lin. <u>dependent</u> for $a = 3/2$.

We have answered the formal question of the problem, but let's look at the answer

Geometrically:

$a = 3/2$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \underline{v}_2 = 2\underline{v}_1$



$\underline{v}_1 \parallel \underline{v}_2$

So: (2 vectors are <u>lin. dep.</u>) $\iff$ (they are parallel)

Note: For 3 (or more) vectors, the criterion of linear dependence is **different** (they don't have to be $\parallel$, but still can be lin. dependent).

We will discuss this criterion next.

Ex. 4 W/o a calculation, show that $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ are lin. <u>dependent</u>.

Sol'n: (Begin by following Ex. 1)

1) $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{\theta}$ (setup to check linear dependence.)

2) Matrix form, by Key Formula:

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underline{\theta}$$

3) Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right)$$

This l.s. is:

$\xleftarrow{\text{Thm. 4}} \begin{cases} \bullet \text{ homogeneous;} \\ \bullet\ 2\times 3,\quad 2<3 \end{cases}$

It has ∞ many sol'ns, $\Rightarrow$ it has nonzero $c_{1,2,3}$.

Then $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are lin. dep. by Def*.

$\boxed{\text{So, in a plane, any 3 vectors are lin. dependent.}}$

This can be generalized:

Thm. 11 Let $\{\underline{v}_1, \dots \underline{v}_n\}$ be a set of vectors in $\mathbb{R}^m$, and let $m < n$. Then these vectors are lin. dependent.

Clarification: This Thm. does not apply when $m \geq n$. I.e., n vectors with $n \leq m$ may or may not be lin. dep.; there is no general conclusion.

E.g., let $m = 3$, $n = 2$.

- Take $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

  They are lin. dep.: $\underline{v}_2 = 2\,\underline{v}_1$.

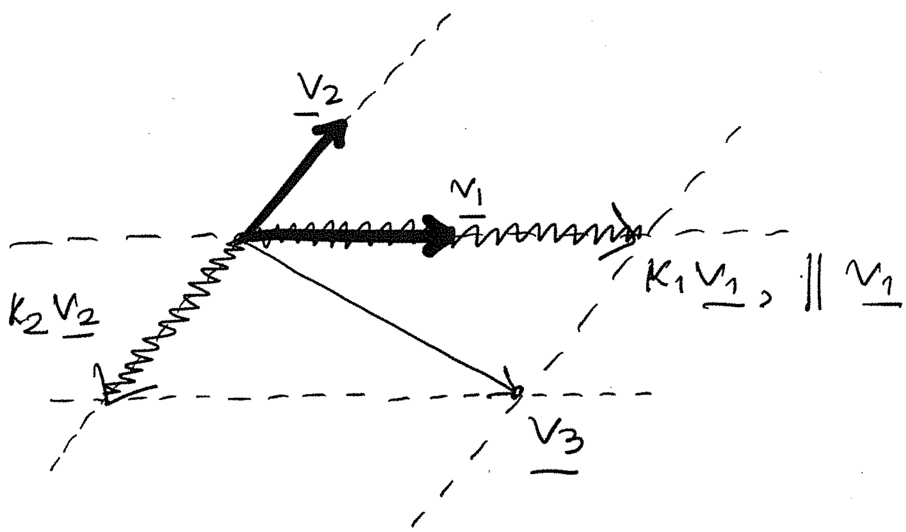
- But, take now $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

  These are clearly lin. indep.

Now, let's look again at 3 vectors in $\mathbb{R}^2$.

Geometrically:

$$\underline{v}_3 = k_1 \underline{v}_1 + k_2 \underline{v}_2 \quad \text{means}$$



"parallelogram"

Practice this concept using the Matlab code posted after these Notes.

So, a **geometric criterion of linear dependence of 3 vectors:**

$$\boxed{\begin{gathered}\textbf{(3 vectors are lin. dependent)} \\ \Longleftrightarrow \\ \textbf{(they lie in the same plane).}\end{gathered}}$$

See the Summary of lin. dependence of 2 & 3 vectors on next page.

Summary of Ex. 3 & 4

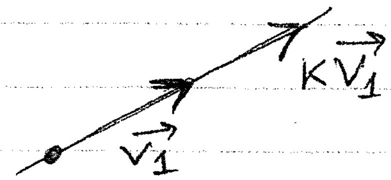
(a) If $\vec{v_2} \parallel \vec{v_1} \Rightarrow$

$\vec{v_2}$ & $\vec{v_1}$ are lin. dependent.

Indeed:

$$\vec{v_2} = k\,\vec{v_1}$$

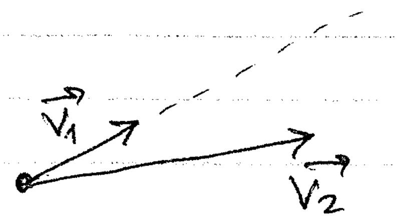
($\vec{v_2}$ depends on $\vec{v_1}$).



(b) If $\vec{v_2} \nparallel \vec{v_1} \Rightarrow$

$\vec{v_2}$ & $\vec{v_1}$ are lin. independent:

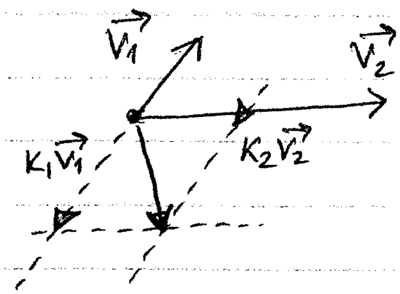
there is no k s.t. $\vec{v_2} = k\vec{v_1}$.



(c) Let $\vec{v_1} \nparallel \vec{v_2}$.

Then any $\vec{v_3}$ in the plane defined by $\vec{v_1}, \vec{v_2}$ is lin. dependent on $\vec{v_1}$ & $\vec{v_2}$:

$$\vec{v_3} = k_1\vec{v_1} + k_2\vec{v_2}$$



(d) If $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are not in the same plane, then they are lin. independent:

$$\vec{v_3} \neq k_1\vec{v_1} + k_2\vec{v_2}$$

for any k_1, k_2.