

Sec. 1.9

8-1

Matrix Inverse & Its Properties

① Motivation

Analogy with scalars :

$$a \cdot \frac{1}{a} = 1, \text{ or } a \cdot a^{-1} = 1$$

When do we usually use "division by a " ?

When solving an equation!

$$ax = b \Rightarrow (\text{if } a \neq 0) \frac{ax}{a} = \frac{b}{a}$$
$$\Rightarrow x = \frac{b}{a}.$$

Another way to write the same :

$$a^{-1} \cdot (ax = b) \Rightarrow \underbrace{(a^{-1}a)}_{\text{"scalar"} \downarrow} x = a^{-1}b \Rightarrow \boxed{x = a^{-1}b}$$

So, we want an analogue of formula (3) for l.s.

② Definition and example of A^{-1}

Def: Let A be $n \times n$. Matrix B is called the inverse of A if:

$$AB = I = BA.$$

Notation: $B \equiv A^{-1}$ ("A inverse")

Ex. 1 Let $A = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}$. Verify that

$$A^{-1} = \frac{1}{3-2a} \cdot \begin{pmatrix} 3 & -2 \\ -a & 1 \end{pmatrix}.$$

Sol'n: It suffices to verify that $A \cdot A^{-1} = I$
(see the definition; $A^{-1} \cdot A = I$ is shown similarly).

$$\text{So, } A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix} \cdot \frac{1}{3-2a} \begin{pmatrix} 3 & -2 \\ -a & 1 \end{pmatrix} =$$

$$= \frac{1}{3-2a} \left(\begin{array}{c|c} 1 \cdot 3 + 2 \cdot (-a) & 1 \cdot (-2) + 2 \cdot 1 \\ a \cdot 3 + 3 \cdot (-a) & a \cdot (-2) + 3 \cdot 1 \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Note 1: You may notice that entries of A^{-1} have some visible relation to entries of A . This is indeed so, but this is true only for 2×2 matrices. See pp. 98-99 of textbook for the formula (optional).

Note 2 Notice that A^{-1} does not exist when $3-2a=0$ (we have then $A^{-1} = \frac{1}{0} \cdot (\dots)$). But, $3-2a=0$, or $a=3/2$, is when A is singular (see Ex. 5 in Sec. 1.7/notes). So, we may suspect that the existence of A^{-1} and A being (non)singular are somehow related.

This agrees with the observation made in Sec. 1.7, that a singular A is the counterpart of the scalar 0 , and we know that 0^{-1} does not exist. We'll soon confirm the suspicion stated above.

③ Usage of A^{-1}

We'll now follow the steps of the derivation of formula (5) on p. 8-1, which was obtained for a scalar equation, and apply it to a l.s. $A\underline{x} = \underline{b}$.

Let's assume that A^{-1} exists, and multiply by it:

$$\begin{aligned}
 & A^{-1} (A \underline{x} = \underline{b}) \\
 & A^{-1} (A \underline{x}) = A^{-1} \underline{b} \\
 \text{I} \quad & \left(\cancel{A^{-1} A} \right) \underline{x} = A^{-1} \underline{b} \\
 & \text{I} \quad \underline{x} = A^{-1} \underline{b} \\
 & \boxed{\underline{x} = A^{-1} \underline{b}} \quad \left(\begin{array}{c} \downarrow \\ \text{"vector"} \\ \underline{V} \end{array} \right)
 \end{aligned}$$

This is the counterpart of (S) that we've sought.

Note 1 You can multiply by A^{-1} (if it exists),
but **you can never divide by A !**

Note 2 Note the multiplication order: we were multiplying on the left. We cannot multiply our l.s. on the right:

$$\begin{array}{ccc}
 (A \underline{x} = \underline{b}) \cdot A^{-1} & & \\
 \begin{array}{cc}
 2 \times 1 & 2 \times 2 \\
 \uparrow & \uparrow
 \end{array} & \longleftarrow & \begin{array}{l} \text{dimensions} \\ \text{do not} \\ \text{match!} \end{array}
 \end{array}$$

Even if we had overlooked the dimensions problem, we would have gotten on the l.h.s.:

$$A \underline{x} A^{-1} \neq A A^{-1} \underline{x}, \text{ because } \underline{x} A^{-1} \neq A^{-1} \underline{x}.$$

Ex. 2 Use the matrix inverse to solve the l.s.:

$$x_1 + 2x_2 = 4$$

$$x_1 + 3x_2 = 5$$

Sol'n: 1) Put this l.s. in matrix form:

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_x = \underbrace{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}_b$$

2) By formula (V) from the previous page:

$$x = A^{-1} b.$$

To find A^{-1} , use the formula from Ex. 1, because

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix} \Big|_{a=1}. \text{ Then: } A^{-1} = \frac{1}{\cancel{3-2 \cdot 1}} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

3) Compute:

$$x = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

④ Existence and uniqueness of the inverse

On p. 8-2 we noted that there may be a link between A having the inverse and A being (non)singular.

We now can state the precise relation.

Thm. 15 $(A^{-1} \text{ exists}) \Leftrightarrow (A = \text{nonsingular})$

a) Proof of " \Rightarrow ".

- Given: A^{-1} exists Want:
- $Ax = \theta \Rightarrow x = \theta$ (Def)
 - Columns of A are l. indep. (Thm. 12)
 - $Ax = b$ has 1 sol'n (Thm. 13)

So: $(A^{-1} \text{ exists}) \Rightarrow (Ax = b \text{ has 1 sol'n}) \Rightarrow (A = \text{nonsing.})$

↑
by formula (V) on p. 8-3

b) Proof of " \Leftarrow ".

Instead of a usual proof, we'll work out an example that shows how A^{-1} can be constructed. We'll then comment where the condition ($A = \text{nonsing.}$) was used.

Ex. 3 Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find A^{-1} .

Sol'n: This $A^{-1} = B$, where, by the Def. in topic (2), B satisfies $AB = I$.

Following the method of Ex. 6/Sec. 1.5, Ex. 8/Sec. 1.7:

let $B = [B_1, B_2]$. Then:

$$AB = I \Rightarrow A[B_1, B_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \underbrace{[AB_1, AB_2]}_{= I} = \underbrace{\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]}_{= I}$$

• $AB_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 0 \end{array} \right) \xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array} \right) \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right) \begin{array}{l} B_1 \\ \downarrow \end{array}$$

• $AB_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & 1 \end{array} \right) \xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array} \right) \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array} \right) \begin{array}{l} B_2 \\ \downarrow \end{array}$$

$$\Rightarrow B = [B_1, B_2] = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \leftarrow \text{Same as in Ex. 2.}$$

Note 1 The condition " $A = \text{nonsing.}$ " was used when we solved to l.s. $AB_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ etc.. By Thm. 13, the fact that $A = \text{nonsing.}$ guaranteed that this l.s. had a (unique) sol'n, \Rightarrow it guaranteed that we could find a unique $B (= A^{-1})$.

Note 2 The steps of the REF work for B_1 & B_2 were exactly the same. Thus, we can combine them:

$$\left[A \mid \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_I \right] \xrightarrow{\text{REF}} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \mid \underbrace{(B_1, B_2)}_{A^{-1}} \right]$$

Thus, the general algorithm is:

$$\left[A \mid I \right] \xrightarrow{\text{REF}} \left[I \mid A^{-1} \right]$$

See Ex. 3(a) and Ex. 2 in book for numeric examples.

Note 3 If you are given some A and follow the above algorithm, but cannot get the "I" on the right-hand side? Then:

- either $A = \text{singular}$, or
 - you've made a mistake.
- ← This can happen on a HW problem, but cannot happen on a quiz or test!

Note 4 (OPTIONAL)

In the Def., it is said that A and A^{-1} commute, which in general is not true for matrices, as we know since Sec. 1.5. So, let's show now that:

$$(AB = I) \Rightarrow (BA = I).$$

(The converse can be established similarly.)

1) let's first show that $B = \text{nonsingular}$.

This follows from Ex. 9 in Sec. 1.7:

$$(AB = I = \text{nonsing.}) \Rightarrow (A \text{ and } B = \text{nonsing.})$$

2) Since $B = \text{nonsing.}$, B^{-1} exists (by Thm. 15).

Let's denote $B^{-1} = C$. Then, by Thm. 15, this C satisfies $BC = I$. Since we want to show that $BA = I$, it remains to show that $C = A$.

3) The calculation:

$$\begin{aligned} (AB = I) \cdot C &\Rightarrow (AB)C = IC \\ \Rightarrow A(BC) = C &\stackrel{2)}{\Rightarrow} A \cdot I = C \Rightarrow A = C. \end{aligned}$$

q.e.d.

⑤ Algebraic properties of the matrix inverse

Thm. 17 Let $A, B = n \times n$ and let A^{-1}, B^{-1} exist.
Then:

1) $(A^{-1})^{-1} = A$ (Basically, we proved this in Note 4)

2) $(AB)^{-1} = B^{-1}A^{-1}$ (note the opposite order!)

3) If $\kappa = \text{scalar} \neq 0$, $\Rightarrow (\kappa A)^{-1} = \frac{1}{\kappa} A^{-1}$

4) $(A^T)^{-1} = (A^{-1})^T$

Proof of 2): It suffices to show that

$(AB) \cdot (B^{-1}A^{-1}) = I$, because this means precisely that $B^{-1}A^{-1} = (AB)^{-1}$ by the Def. in topic ②.

$$\begin{aligned} \text{So: } (AB) \cdot B^{-1}A^{-1} &= A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} \\ &= A \cdot A^{-1} = I. \quad \checkmark \end{aligned}$$

Q: Why wouldn't $A^{-1}B^{-1}$ work? Try it:

$$\begin{aligned} (AB)(A^{-1}B^{-1}) &= A \underbrace{(BA^{-1})} B^{-1} \Rightarrow \text{cannot combine} \\ &\neq A^{-1}B \quad \text{A \& A^{-1} and B \& B^{-1} \Rightarrow stuck...} \end{aligned}$$

Proof of 4) (OPTIONAL)

We need to show that $(A^T)^{-1} = (A^{-1})^T$,
i.e. (by the Def. in topic ②), we want:

$$A^T \cdot (A^{-1})^T = I. \quad \text{Compute the l.h.s.?:}$$

$$A^T \cdot (A^{-1})^T \stackrel{\substack{\uparrow \\ \text{Thm. 10}}}{=} (A^{-1} \cdot A)^T = I^T = I. \quad \checkmark$$

Thm. 18 (Summary of main results of Secs. 1.7, 1.9)

Let A be $n \times n$. The following statements are equivalent:

1. A = nonsingular (Def. in 1.7-B)
2. Columns of A are lin. indep. (Thm. 12)
3. $A\underline{x} = \underline{b}$ has 1 sol'n (Thm. 13)
4. A^{-1} exists. (Thm. 15)
5. The REF of A is I . (used only in computation, not for proofs)

⑥ More examples of proofs

Ex. 4. In Sec. 1.6 we encountered a strange property of matrices:

$$\left(\begin{array}{l} AB = AC \\ \text{and} \\ A \neq \mathcal{O} \end{array} \right) \not\stackrel{\text{in general}}{=} (B = C).$$

We can now modify this statement to have " \Rightarrow " hold:

$$\left(\begin{array}{l} AB = AC \\ \text{and} \\ A = \text{nonsing.} \end{array} \right) \Rightarrow (B = C).$$

Proof: Given:

- 1) $AB = AC$
- 2) One of the 5 statements in Thm. 18 (The Summary)

Want:

$$B = C$$

We need to decide which statement in Thm. 18 to use. We are in the section on matrix inverses \Rightarrow the common sense suggests that we use " A^{-1} exists".

$$\begin{aligned} \text{Then: } AB = AC &\Rightarrow A^{-1} \cdot (AB = AC) \\ &\Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I \cdot B = I \cdot C \\ &\Rightarrow B = C. \quad \checkmark \end{aligned}$$

Note: Let us repeat the point made in Note 2 on p. 8-3: we must multiply by A^{-1} on the left.

Indeed, multiplication on the right would not work:

$$\begin{aligned} (AB)A^{-1} &= (AC)A^{-1} \Rightarrow A(BA^{-1}) = A(CA^{-1}) \\ &\not\Rightarrow AA^{-1}B = AA^{-1}C, \text{ because} \\ &BA^{-1} \neq A^{-1}B \text{ and } CA^{-1} \neq A^{-1}C. \end{aligned}$$

Ex. 5. In Ex. 9 in Sec. 1.7 we showed

$$\boxed{(A \text{ or } B = \text{singular}) \Rightarrow (AB = \text{singular})} \quad (1a)$$

or, equivalently (by negation),

$$\boxed{(A \text{ and } B = \text{nonsingular}) \Leftrightarrow (AB = \text{nonsingular})} \quad (1b)$$

Let's now prove the converse of (1b):

$$(A \text{ and } B = \text{nonsingular}) \Rightarrow (AB = \text{nonsingular}) \quad (2b)$$

((2a) is not written; it is meant to be converse of (1a).)

Proof: Again, we are in the section about the matrix inverse, so it is the common sense to use statement 4 of Thm. 18:

$$(A \& B = \text{nonsing.}) \xrightarrow{\text{Thm. 18}} (A^{-1} \& B^{-1} \text{ exist})$$

$$\xrightarrow{\text{Thm. 17-2}} ((AB)^{-1} \text{ exists}) \xrightarrow[\text{or Thm. 18-4}]{\text{Thm. 15,}} (AB = \text{nonsing.})$$

q.e.d.

So, combining (2b) & (1b):

$$\boxed{(A \& B = \text{nonsing.}) \Leftrightarrow (AB = \text{nonsing.})} \quad (2)$$

by negation:

$$\boxed{(A \text{ or } B = \text{sing.}) \Leftrightarrow (AB = \text{sing.})} \quad (1)$$

This should make intuitive sense if we recall the earlier-stated analogy:

(a singular matrix) is the analogue of (number 0)

Then (1) is the matrix analogue of the statement:

$$(\text{numbers } a \& b \neq 0) \Leftrightarrow (ab \neq 0).$$