

Sec. 3.1. Introduction to vector spaces \mathbb{R}^n :
 \mathbb{R}^2 and \mathbb{R}^3

In this Chapter we'll study so-called vector spaces \mathbb{R}^n and common properties of these spaces.

We'll introduce a definition of a vector space later. For now it will suffice to know that \mathbb{R}^n is the set of all vectors with n real components.

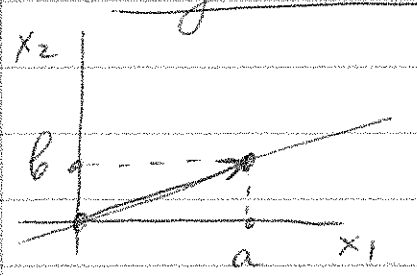
If you have difficulty imagining a particular situation in \mathbb{R}^n — think how it would work in \mathbb{R}^2 and \mathbb{R}^3 (the plane and the 3D space).

Most properties of \mathbb{R}^n can be understood by examples in \mathbb{R}^2 and \mathbb{R}^3 .

notation $\rightarrow \mathbb{R}^2 = \{x : x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 = \text{real}\}$

Below is a review from Calculus III.

① Eq. of a line in \mathbb{R}^2 that goes through $(0,0)$ along vector $\langle a, b \rangle$:



$x_1 = at$
 $y_1 = bt$
(Check: $y = bt = b \cdot \frac{x}{a} = (\frac{b}{a})x$. \checkmark)

Notation:

$\{x : x = \begin{pmatrix} at \\ bt \end{pmatrix}, t = \text{real}\}$

② An alternative form of ①.

If $x_1 = at$, $x_2 = bt$, then

$$(-b) \cdot (at) + a \cdot (bt) = 0$$

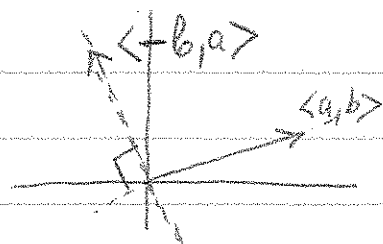
$$\underbrace{-b}_{A} \cdot x_1 + \underbrace{a}_{B} \cdot x_2 = 0$$

call this

$$Ax_1 + Bx_2 = 0$$

$$\begin{pmatrix} -b \\ a \end{pmatrix}^T \underline{x} = 0$$

The line is \perp to vector $\begin{pmatrix} -b \\ a \end{pmatrix} \equiv \begin{pmatrix} A \\ B \end{pmatrix}$.

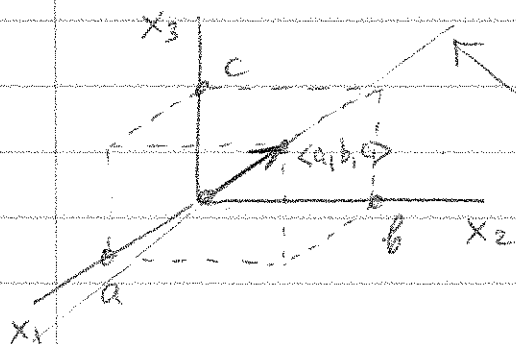


Eq. of a line in \mathbb{R}^2 that goes through $(0,0)$.

③ Eq. of a line in \mathbb{R}^2 not through origin:

$$Ax_1 + Bx_2 + C = 0, \quad C \neq 0.$$

④ Eq. of a line in \mathbb{R}^3 going through $(0,0,0)$ and along vector $\langle a, b, c \rangle$:



$$\left\{ \underline{x} : \underline{x} = \begin{pmatrix} at \\ bt \\ ct \end{pmatrix}, t = \text{real} \right\}$$

Note: the eq. of a line in 3D cannot be written as $Ax_1 + Bx_2 + Cx_3 = 0$.

In fact, we need two equations, not one, to define a line in \mathbb{R}^3 . See Ex. 2 below.

⑤ Eq. of a plane (in \mathbb{R}^3) that is \perp to vector $\langle A, B, C \rangle$:

$$\boxed{Ax_1 + Bx_2 + Cx_3 = D} \quad (\star)$$

If $D = 0$, the plane goes through the origin.

Ex. 1 What is $\{x: x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_2 = x_1\}$?

Sol'n: Take the constraining equation:
 $x_2 = x_1$. Put it in form \star :

$$(-1)x_1 + 1x_2 + 0x_3 = 0$$

So this is a plane $\perp \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. $D=0$, \Rightarrow it goes through the origin.

Geometric interpretation: count "degrees of freedom".

$$\begin{array}{ccc} 3 & - 1 & = 2 \\ \text{(in } \mathbb{R}^3 \text{ we} & \text{(impose 1 eq} & \text{DoFs remain.} \\ \text{have 3 DoF)} & \text{= 1 constraint)} & \end{array}$$

A plane has 2 DoFs, but a line has only 1 DoF.

Ex. 2 What is $\{x: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_1 = x_2 \text{ and } x_2 = -x_3\}$?

Sol'n: Write each constraining eq. in form \star :

$$\text{Ex. 1} \Rightarrow -1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = 0 \leftarrow \text{plane}$$

&

$$0 \cdot x_2 + 1 \cdot x_2 + 1 \cdot x_3 = 0 \leftarrow \text{another, non-parallel, plane.}$$

The "and" says that x belongs to both planes,
 \Rightarrow it must be ~~along~~ ^{on} the intersection line
of these two planes.

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Thus, the required set is a line in 3D.

Observation: A line in 3D is defined by 2,
not 1, equations.

Reason: count DoFs.

$$\begin{array}{rcl} 3 & - & 2 & = & 1 \\ \text{(3 DoFs} & & \text{(impose 2} & & \text{DoF along} \\ \text{in } \mathbb{R}^3\text{)} & & \text{constraining eqs.)} & & \text{a line} \end{array}$$

HW: 5, 7, 13, 14, 15, 16, 18, 19, 20, 23, 25,
27, 28, 29, 30,
+
2 Word Problems (online).