

Sec. 3.2: Vector space properties of \mathbb{R}^n

10-1

① Properties of \mathbb{R}^n

Thm. 1 If $\underline{x}, \underline{y}$ are any vectors in \mathbb{R}^n and c is any scalar, then the following groups of properties hold:

(A) ← addition of vectors
(M) ← multiplication by a scalar

} similar to corresponding properties of scalars, but
SEE p. 168 in book.

(C) ← closure:

$$(C1) (\underline{x}, \underline{y} \text{ in } \mathbb{R}^n) \Rightarrow (\underline{x} + \underline{y} \text{ in } \mathbb{R}^n)$$

$$(C2) (\underline{x} \text{ in } \mathbb{R}^n) \Rightarrow (c \cdot \underline{x} \text{ in } \mathbb{R}^n)$$

Def: Any set of quantities that satisfies properties (A), (M), (C) is called a vector space.

This Def. & Thm. 1 imply:

Any set of quantities which satisfies (A), (M), (C) "behaves" exactly as vectors in \mathbb{R}^n . (And, since vectors in \mathbb{R}^n behave as vectors in \mathbb{R}^2 and \mathbb{R}^3 , one can visualize any vector space by visualizing what goes on in \mathbb{R}^2 and \mathbb{R}^3 .)

Here are two examples of vector spaces:

Ex. 1 $\mathcal{P} = \{ p_n(x) \},$

the set of all polynomials of x of degree $\leq n$.

(Indeed, properties (A) & (M) are verified straight forwardly (once you look at p.168); then it remains to check the closure properties.

(C1): adding 2 polynomials of degree $\leq n$, you always get a polynomial of degree $\leq n$.

(C2): multiplying a polynomial of degree $\leq n$ by a scalar (say, 3), you get a polynomial of degree $\leq n$.)

Ex. 2 The set of all continuous functions is a vector space.

(Indeed, the sum of two continuous functions is a continuous function (C1), and if you multiply a continuous function by scalar C , you get a continuous function.)

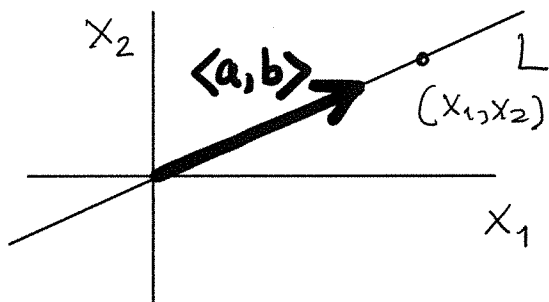
Note: In Project 2 you see an explicit relation between continuous functions $f(x)$ and vectors in \mathbb{R}^n : You approximate the elevation of a cable (a continuous function) by an ordered set of its values (y_1, y_2, \dots, y_n) at discrete points (x_1, x_2, \dots) .

• Yet another example of a vector space is the set of solutions of a linear homogeneous differential equation (**MATH 271**), and there are many more

Let us now look at two examples of subsets of \mathbb{R}^2 , one of which is a vector space, and the other is not a vector space.

Ex. 3 A line passing through the origin in \mathbb{R}^2 is a vector space.

Method 1 - algebraic (it is still a good idea to make a sketch)



The fact that point (x_1, x_2) is on L is equivalent to the fact that vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$ are \parallel :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \parallel \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \text{topic 2 in Sec. 3.1}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t \text{ for some } t.$$

Then:

1) Properties (A) & (M) hold for all vectors (but you still need to look at p. 168!), and so you will never need to check (A) & (M).

So it remains to check (c1) & (c2).

2) Checking (c1): $(\underline{x}, \underline{y} \text{ in } L) \xrightarrow{?} (\underline{x} + \underline{y} \text{ in } L)$

$$\underline{x} \text{ in } L \Rightarrow \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_1 \text{ for some } t_1$$

$$\underline{y} \text{ in } L \Rightarrow \underline{y} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_2 \text{ for some } t_2$$

So, is $(\underline{x} + \underline{y}) = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_3$ for some t_3 ?

Check this: $\underline{x} + \underline{y} = \begin{pmatrix} a \\ b \end{pmatrix} t_1 + \begin{pmatrix} a \\ b \end{pmatrix} t_2 =$
 $= \begin{pmatrix} a \\ b \end{pmatrix} \cdot (t_1 + t_2) \rightarrow$ call this "t₃".

here we have used some properties from p. 168 ...

So, yes, indeed, (c1) holds. ✓

3) Checking (c2): (\underline{x} in L) $\stackrel{?}{\implies}$ ($c \cdot \underline{x}$ in L)

\underline{x} in L $\implies \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} t_1$ for some t_1 .

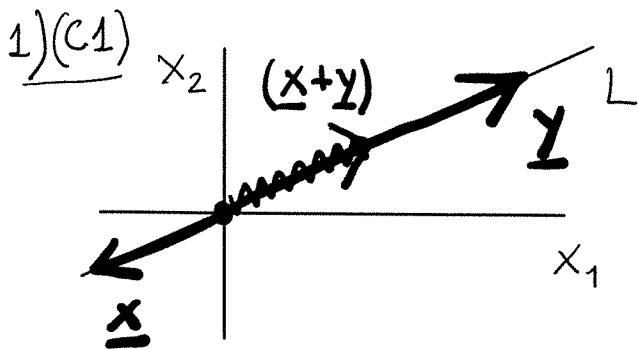
So, is $c \cdot \underline{x} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_2$ for some t_2 ?

Check this: $c \underline{x} = c \cdot \begin{pmatrix} a \\ b \end{pmatrix} \cdot t_1 = \begin{pmatrix} a \\ b \end{pmatrix} \cdot (ct_1)$ call this "t₂".

So, yes, indeed, (c2) holds. ✓

Method 2 - geometric

We still need to check the same two properties (c1) & (c2), but now we do so with a sketch.

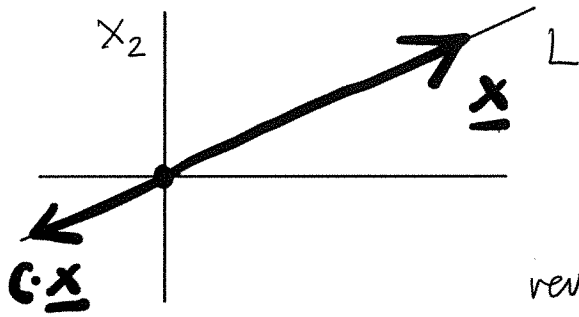


• Take two vectors \underline{x} & \underline{y} such that their end points are on L. (Recall that the starting point of a vector is always at the origin, by def.)

• Draw $(\underline{x} + \underline{y})$. From Calculus, recall that when you add two parallel (or anti-parallel) vectors, you'll get a vector parallel to the original ones. So, as shown above, $(\underline{x} + \underline{y})$ is also on L, \implies (c1) holds. ✓

Note: You didn't need to take \underline{x} and \underline{y} anti-parallel, as above. You could have taken them to be parallel. $(\underline{x} + \underline{y})$ would then be longer than both \underline{x} & \underline{y} , but it will still be on L , and so the same conclusion as above would hold.

2) (c2)



- Take some \underline{x} on L .
- Draw $c \cdot \underline{x}$. Again, from Calculus, $c \cdot \underline{x} \parallel \underline{x}$ (we also reviewed this in Secs. 1.7 & 3.1).

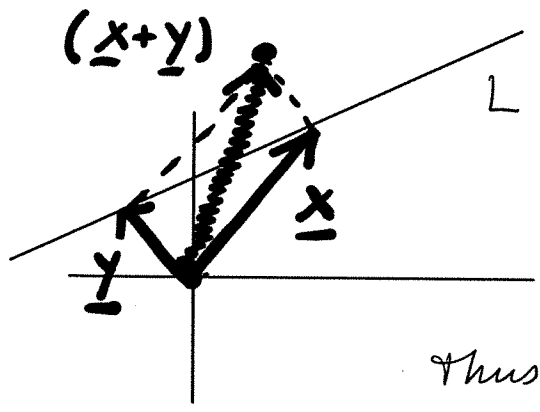
So you get $c \cdot \underline{x}$ as shown. (In this particular example, $c < 0$ and $|c| < 1$ — make sure you understand why; if you don't — review the very first section about vectors in your Calculus text.)

The important point is that $c \cdot \underline{x}$ is on L , \Rightarrow (c2) holds.

Note: You may notice that Method 2 (geometric) is easier to apply than Method 1 (algebraic); at least it requires less writing. Therefore, I strongly recommend that you use the geometric method whenever possible. (Using the algebraic method is OK, too, if you are more comfortable with it.)

Ex. 4 A line not passing through the origin in \mathbb{R}^2 is not a vector space.

Sol'n: Use the geometric method.



Check (c1):

Take \underline{x} and \underline{y} on L. Then $(\underline{x+y})$ is not on L.

Thus, (c1) FAILS, and there is no need to check (c2). L is not a vec. space.

② Subspaces of \mathbb{R}^n

Note: A vector space W has properties (c1) & (c2):

- (c1) $(\underline{x}, \underline{y} \text{ in } W) \Rightarrow (\underline{x+y} \text{ in } W)$
- (c2) $(\underline{x} \text{ in } W) \Rightarrow (c \cdot \underline{x} \text{ in } W \text{ for any } c).$

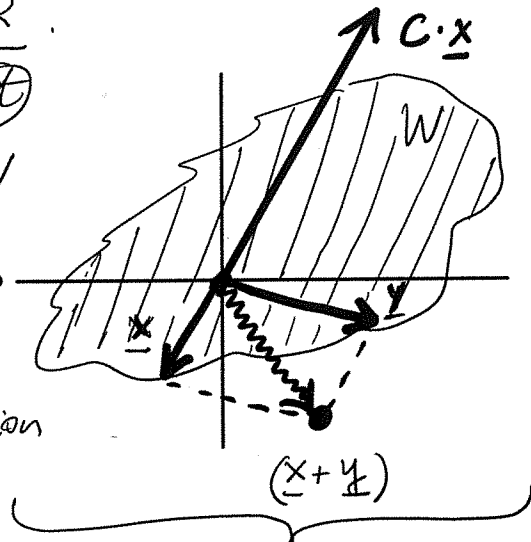
combined: $(\underline{x}, \underline{y} \text{ in } W) \Rightarrow (\underbrace{(a\underline{x} + b\underline{y})}_{\substack{\text{linear} \\ \text{combination} \\ \text{of } \underline{x} \ \& \ \underline{y}}} \text{ in } W \text{ for any } a, b)$

In this course we'll be working with subsets W of \mathbb{R}^n and will need to take linear combinations of vectors in W. We will want these linear combinations to remain in W. Thus, we'll want these subsets W to be vector spaces themselves.

Def: A subset W of \mathbb{R}^n such that W itself is a vector space is called a **subspace of \mathbb{R}^n** .

Thus, we want to work only with those W which are subspaces of \mathbb{R}^n .

In other words, we do not want to work with such W where for some \underline{x} & \underline{y} in W , either $(\underline{x} + \underline{y})$ or $c \cdot \underline{x}$, or some other linear combination $(a \cdot \underline{x} + b \cdot \underline{y})$,



may be not in W . Example of a bad W .

To check if a given W is a subspace, you need to check (algebraically or graphically) if (c1) & (c2) hold for W :

$$(W \text{ is subspace of } \mathbb{R}^n) \iff (c1), (c2) \text{ hold for } W.$$

- Example of a subspace of \mathbb{R}^2 : line through origin (Ex. 3).
- Example of a non-subspace of \mathbb{R}^2 : line not through origin (Ex. 4).

Trivial subspaces of \mathbb{R}^2 (or any \mathbb{R}^n): $\{\underline{0}\}$, \mathbb{R}^n .

Name: A subspace of \mathbb{R}^n that is neither $\{\underline{0}\}$ nor \mathbb{R}^n is called a proper subspace.

- How to quickly check if W is not a subspace:

Note that if W is a subspace, then any lin. combination $(ax + by)$ is in W . Take $a = b = 0$. Then $0 \cdot x + 0 \cdot y = \underline{0}$ is in W . So:

$$(W \text{ is a subspace of } \mathbb{R}^n) \Rightarrow (\underline{0} \text{ is in } W).$$

Take the contrapositive:

$$(\underline{0} \text{ is } \text{not} \text{ in } W) \Rightarrow (W \text{ is } \text{not} \text{ a subspace}).$$

So, the QUICK CHECK is:

- $(\underline{0} \text{ not in } W) \Rightarrow (W \text{ not a subspace of } \mathbb{R}^n)$
- $(\underline{0} \text{ in } W) \Rightarrow \dots$ **more work needed to determine if W is a subspace of \mathbb{R}^n .**

③ Examples of subspaces & non-subspaces in \mathbb{R}^2 and \mathbb{R}^3 .

In Ex. 3 we saw that a straight line through the origin in \mathbb{R}^2 is a subspace of \mathbb{R}^2 .

In Ex. 4 — " — " — " — not through — " — " — " — is not a subspace of \mathbb{R}^2 .

This can be generalized:

Claim 1: In \mathbb{R}^2 , the only proper subspace is any straight line through the origin.

So, on quiz or test, you may answer the question of whether some W is a subspace of \mathbb{R}^2 as follows:

1. Use Sec. 3.1 to decide if W is a straight line through the origin in \mathbb{R}^2 .
2. Use Claim 1.

Claim 2: In \mathbb{R}^3 , the only proper subspaces are:

- any straight line through the origin;
- any plane through the origin.

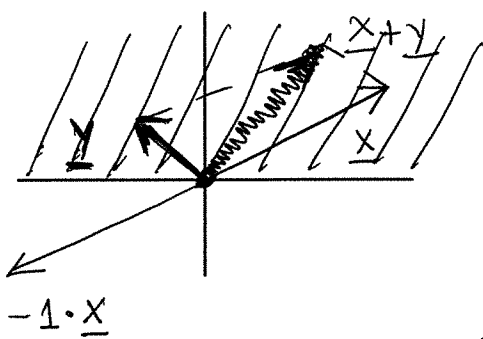
So, on a quiz or test, you may answer the question if some W is a subspace of \mathbb{R}^3 as follows:

1. Use Sec. 3.1 to decide if W is a line or plane through the origin.
2. Use Claim 2.

MUST READ Ex. 1, 2 in book / 3.2 about this.

Now, let us look at examples of non-subspaces of W (the goal being to see how the closure properties (c1), (c2) can fail, not to just apply Claim 1).

Ex. 5 Upper-half plane ($x_2 \geq 0$) is not a subspace.



Check (c1):

$\underline{x}, \underline{y}$ in UHP (point up),
 $\Rightarrow (\underline{x} + \underline{y})$ in UHP (points up).

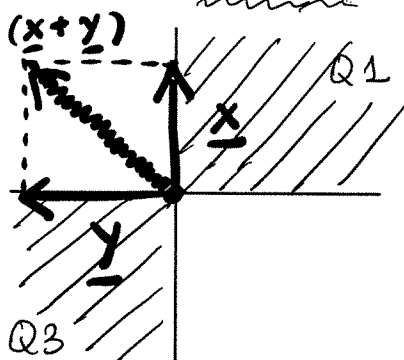
✓

Check (c2):

$-1 \cdot \underline{x}$ not in UHP.

So, (c2) FAILS, \Rightarrow UHP is not a subspace. //

Ex. 6 Union of Quadrants 1 and 3 :



$$W = \{ \underline{x} : \underline{x} \text{ in } Q1 \text{ or } \underline{x} \text{ in } Q3 \}$$

Check (c1):

$\underline{x}, \underline{y}$ in W , but
 $(\underline{x} + \underline{y})$ not in W .

Thus, (c1) FAILS, and there is no need to check (c2).
 So, the union of $Q1$ & $Q3$ is not a subspace of \mathbb{R}^2 .

Note: Note that in both Ex. 5 & 6,
 $\underline{0}$ was in W . However, this didn't make W a
 subspace. See again the QUICK CHECK on p. 108.

MUST READ: Ex. 3 & 5 in book (3.2).

In Ex. 5, pay attention to word "or".

Compare it and word "and" in Ex. 2 in Notes
 for Sec. 3.1.

④ A simple example of why we care
whether some W is or is not a vector space
(or a subspace of \mathbb{R}^n). (OPTIONAL)

homogeneous

In Sec. 3.3 we'll show that all solutions of
 a l.s. form a subspace of \mathbb{R}^n . (You've already
 seen, and will continue to see in this course and
 in MATH 271 that homogeneous l.s. are important.)

Here we'll illustrate, by a simple example, that solutions of only homogeneous AND linear systems form vector spaces. In other words (recall p. 10-6), only solutions of homog. lin. systems satisfy the superposition principle: (if $\underline{x}, \underline{y}$ are solutions), then $(a\underline{x} + b\underline{y})$ is also a sol'n for any a, b).

Ex. 7(a) System is linear and homogeneous:

$$x - y = 0$$

Its solution: $x = y, y = \text{free}$, i.e. any vector $\underline{v} = \begin{pmatrix} y \\ y \end{pmatrix}$.

Take some two solutions, say $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Take their linear combination:

$$a\underline{v}_1 + b\underline{v}_2 = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} a+2b \\ a+2b \end{pmatrix} \equiv \begin{pmatrix} c \\ c \end{pmatrix};$$

this is also a sol'n of our l.s.

So, sol'n's of a homog. l.s. obey the linear superposition principle.

Ex. 7(b) System is linear, but non-homogeneous:

$$x - y = 1.$$

Its sol'n: $x = y + 1, y = \text{free}$, or $\underline{v} = \begin{pmatrix} y+1 \\ y \end{pmatrix}$.

Take two sol'n's: $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Take their linear superposition

(a simple sum): $\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

It does not fit the pattern $\begin{pmatrix} y^2+1 \\ y \end{pmatrix}$ and hence is not a sol'n of the l.s..

Thus, sol'ns of a linear, but non-homogeneous systems do not satisfy the lin. superposition principle (a.k.a. „are not vector spaces“).

Ex. 7(c) System is non-linear, and homogeneous:

$$x - y^2 = 0.$$

Take two sol'ns: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Take their sum: $\underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. This doesn't fit the pattern $\begin{pmatrix} y^2 \\ y \end{pmatrix}$ and hence is not a sol'n of the given system.

Thus, only sol'ns of linear & homogeneous systems satisfy the linear superposition principle.