Sec. 4.1 The Eigenvalue Problem for 2×2 matrices

Plan: ① Motivation
② Eigenvalues & eigenvectors of a 2×2 matrix.

① Motivation
In the first lecture, we considered an example about newspaper subscription in a small town.

Ex. 1

Recall that in that town, people either subscribe to the only local newspaper, or they don't, and every year the # of subscribers, S, and # of non-subscribers, N, change as follows:

$$S_1 = 0.7\,S_0 + 0.5\,N_0$$
$$N_1 = 0.3\,S_0 + 0.5\,N_0$$

$$\begin{pmatrix} S \\ N \end{pmatrix}_1 = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} S \\ N \end{pmatrix}_0 \equiv A \begin{pmatrix} S \\ N \end{pmatrix}_0$$

The question we posed then is what happens to the subscribers and nonsubscribers in 20 years, assuming that the trend is preserved.

Obviously, $\begin{pmatrix} S \\ N \end{pmatrix}_2 = A \begin{pmatrix} S \\ N \end{pmatrix}_1 = AA \begin{pmatrix} S_0 \\ N_0 \end{pmatrix} = A^2 \begin{pmatrix} S \\ N \end{pmatrix}_0$,

$$\begin{pmatrix} S \\ N \end{pmatrix}_{20} = A^{20} \begin{pmatrix} S \\ N \end{pmatrix}_0 .$$

How can we find the answer w/o computing A^{20}?

Steps of solution:

1) <u>Assume</u>, for now, that there are such 2 vectors $\underline{v_1}$ and $\underline{v_2} \in R^2$

that: $A\underline{v_1} = \lambda_1 \underline{v_1}$, $A\underline{v_2} = \lambda_2 \underline{v_2}$,

i.e. the effect of multiplying $\underline{v_1}$ by the matrix A is the same as multiplying v_1 by a scalar λ_1. A similar Example was in the HW for sec. 1.6. (#43)

2) Now assume that $\underline{v_1}, \underline{v_2}$ form a basis in R^2, then any vector $\binom{S}{N}_0 = c_1 \underline{v_1} + c_2 \underline{v_2}$

{use the concept of basis →

3) After 1 year $\binom{S}{N}_1 = A\binom{S}{N}_0 = A(c_1\underline{v_1} + c_2\underline{v_2}) = c_1 A\underline{v_1} + c_2 A\underline{v_2} =$

$$= c_1\lambda_1\underline{v_1} + c_2\lambda_2\underline{v_2}.$$

After 2 years:

$$\binom{S}{N}_2 = A^2\binom{S}{N}_0 = A\left(A\binom{S}{N}_0\right) = A\left((c_1\lambda_1)\underline{v_1} + (c_2\lambda_2)\underline{v_2}\right) =$$

$$= (c_1\lambda_1)A\underline{v_1} + (c_2\lambda_2)A\underline{v_2} =$$

$$= (c_1\lambda_1)\lambda_1\underline{v_1} + (c_2\lambda_2)\lambda_2\underline{v_2} = c_1\lambda_1^2\underline{v_1} + c_2\lambda_2^2\underline{v_2}.$$

Etc., $\Rightarrow A^{20}\binom{S}{N}_0 = c_1\lambda_1^{20}\underline{v_1} + c_2\lambda_2^{20}\underline{v_2}$

Note the simplification:

Instead of computing A^{20}, we only need to compute λ_1^{20} and λ_2^{20}!

Ex. 1 adjourned

(2) Def. (The Eigenvalue problem)

Let A be $(n \times n)$, and let there exists a vector $\underline{x}$ and a scalar λ s.t.

$$A\underline{x} = \lambda\underline{x}.$$

Then this λ is called an eigenvalue of A and the nonzero vector $\underline{x}$ is called an eigenvector corresponding to the eigenvalue λ.

Note: We require $\underline{x} \neq \underline{\theta}$, because otherwise

$$A\underline{\theta} = \lambda\underline{\theta}$$

for any # λ.

Consider $A\underline{x} = \lambda\underline{x} \Rightarrow A\underline{x} - \lambda\underline{x} = \underline{\theta}$

$$\overset{?}{\Rightarrow} \; ? \; (A - \lambda)\underline{x} = \underline{\theta} \leftarrow \text{no!} \leftarrow \text{cannot subtract a scalar } \lambda \text{ from matrix } A$$

$$\Rightarrow \; ! \; \boxed{A\underline{x} - \lambda I\underline{x} = \underline{\theta}} \leftarrow \text{Since } \underline{x} = I\cdot\underline{x} \text{ for any } \underline{x}$$

$$\Rightarrow \boxed{(A - \lambda I)\underline{x} = \underline{\theta}}$$

Thus, to find eigenvalues and eigenvectors of A, we perform two steps:

Step 1: Find all λ s.t. $(A - \lambda I)$ = singular

Step 2: Determine the null space of $(A - \lambda I)$, i.e. all $\underline{x}$ s.t. $(A - \lambda I)\underline{x} = \underline{\theta}$.

For the purposes of Ex. 1, A is (2×2), so we consider the case of a general (2×2) matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}.$$

$\big((A - \lambda I) = \text{singular}\big) \Leftrightarrow$ (columns of $(A - \lambda I)$ are lin. dep., i.e. proportional to each other)

$$\Leftrightarrow \frac{a-\lambda}{c} = \frac{b}{d-\lambda} \Rightarrow$$

$$(a-\lambda)(d-\lambda) = bc \Rightarrow ad - a\lambda - d\lambda + \lambda^2 = bc$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

Thus, $(A-\lambda I)$ is singular iff

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

Ex. 1 (continued)

$$A = \begin{pmatrix} \overset{=a}{0.7} & \overset{=b}{0.5} \\ \underset{=c}{0.3} & \underset{=d}{0.5} \end{pmatrix}$$

Finding e-values & e-vectors of A:

Step 1: Find all λ s.t. $(A-\lambda I)$ = singular.

$$\lambda^2 - (0.7+0.5)\lambda + (0.7\cdot 0.5 - 0.5\cdot 0.3) = 0$$

$$\lambda^2 - 1.2\lambda + 0.2 = 0, \quad (\lambda-1)(\lambda-0.2) = 0.$$

see middle of p. 17-3

$$\lambda_1 = 1, \quad \lambda_2 = 0.2$$

Step 2 For each λ above, find $\underline{x}$ s.t. $(A-\lambda I)\underline{x} = \underline{\theta}$

$\lambda = \lambda_1 = 1$:

$A\underline{v}_1 = \lambda_1 \underline{v}_1$:

$$\begin{pmatrix} 0.7-1 & 0.5 \\ 0.3 & 0.5-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -0.3 & 0.5 & 0 \\ 0.3 & -0.5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -0.3 & 0.5 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = \frac{5}{3}x_2 \Rightarrow \underline{V_1} = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} x_2, \quad x_2 = \text{free}$$

$\lambda = \lambda_2 = 0.2$:

$A\underline{v_2} = \lambda_2 v_2$:

$$\begin{pmatrix} 0.7-0.2 & 0.5 \\ 0.3 & 0.5-0.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\left[\begin{array}{cc|c} 0.5 & 0.5 & 0 \\ 0.3 & 0.3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x_1 = -x_2 \Rightarrow \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2, \quad x_2 = \text{free}.$

Thus, $\quad A\underline{\tilde{v}_1} = \lambda_1 \underline{\tilde{v}_1}, \quad \underline{\tilde{v}_1} = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} a, \quad a = \text{free}$

$\qquad A\underline{\tilde{v}_2} = \lambda_2 \underline{\tilde{v}_2}, \quad \underline{\tilde{v}_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} b, \quad b = \text{free}$

We have solved the eigenvalue problem for A.

Finish up Ex. 1.

Note: an eigenvector is always found up to an arb. constant c

E.g.: $A\underline{v} = \lambda \underline{v} \iff c\, A\underline{v} = c\lambda \underline{v} \iff_{(c \neq 0)} A(c\underline{v}) = \lambda(c\underline{v}) \iff c\underline{v} = \text{eigenv.}$

$\underline{v_1} = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{v_1}, \underline{v_2}$ are lin. indep. (by inspection).

$\Rightarrow \begin{pmatrix} S \\ N \end{pmatrix}_0 = c_1 \underline{v_1} + c_2 \underline{v_2} = c_1 \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\left(\text{E.g.,} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{8}\begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + \frac{3}{8}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right).$

Then $\quad \begin{pmatrix} S \\ N \end{pmatrix}_{20} = c_1 \lambda_1^{20} \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \lambda_2^{20} \begin{pmatrix} 1 \\ -1 \end{pmatrix} =$

$\qquad = c_1 \cdot 1^{20} \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \cdot 0.2^{20} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Thus, $\quad \begin{pmatrix} S \\ N \end{pmatrix}_{20} \approx c_1 \begin{pmatrix} 5/3 \\ 1 \end{pmatrix}, \Rightarrow \frac{S_{20}}{N_{20}} \approx 5/3 \; !$

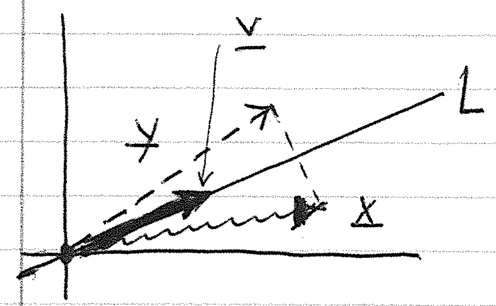
The issues we have exposed but ~~not~~ did not resolve, are:

(1) Can we be sure that for another matrix A, $(\underline{v_1}, \underline{v_2})$ will always form a basis?

(2) How do we generalize this for an $(n \times n)$ matrix? That is, how do we find λ and $\underline{x}$ in practice, if A is (3×3) or $(n \times n)$? $n > 3$

3 Geometric meaning of eigenvectors

Ex. 2(a) Show that the matrix of reflection about a line in $\mathbb{R}^2$ always has an eigenvalue $\lambda = 1$.



Sol'n: Let A be the matrix of our l.t. T.

1) For any vector $\underline{x}$: $T(\underline{x}) = \underline{y}$ ← some $\underline{y}$ (see picture)

$\Rightarrow \boxed{A\underline{x} = \underline{y}}$.

2) For the eigenvector $\underline{v}$: $A\underline{v} = \lambda \underline{v}$ ← for some λ

$\boxed{A\underline{v} = 1 \cdot \underline{v}}$ ← for $\lambda = 1$

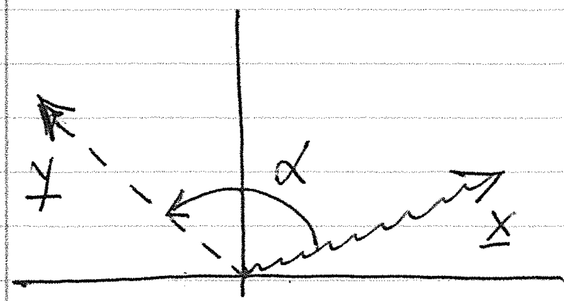
3) Combine 1) & 2): $\boxed{T(\underline{v}) \underset{\substack{\uparrow \\ \text{for any } \underline{v}}}{=} A\underline{v} \underset{\substack{\uparrow \\ \text{for this } \underline{v}}}{=} 1 \cdot \underline{v}}$, i.e.

$\boxed{T(\underline{v}) = \underline{v}}$ ← Thus, if $\underline{v}$ is an eigenvector of the the reflection matrix A with $\lambda = 1$, then the reflection does not change $\underline{v}$!

4) Looking at the figure, we see that such a vector indeed exists: it is any vector along the reflection line L.

Thus, we have found a vector $\underline{v}$ that satisfies $T(\underline{v}) = \underline{v}$. Then, tracing back the steps: $T(\underline{v}) = \boxed{A\underline{v} = \underline{v}}$, we see that this $\underline{v}$ is an eigenvector whose eigenvalue $\lambda = 1$.

<u>Ex. 2(b)</u> Show that the matrix of a nonzero rotation (by angle $\alpha \neq 0, 360^\circ$, etc.) does not have an eigenvalue $\lambda = 1$.



Sol'n: We closely follow the steps of Ex. 2(a). Let A be the matrix of rotation.

1) For any vector $\underline{x}$: $T(\underline{x}) = \underline{y}$ ← some $\underline{y}$; see picture

$\Rightarrow A\underline{x} = \underline{y}$.

2) For an eigenvector $\underline{v}$: $A\underline{v} = \lambda \cdot \underline{v}$ ← for some λ

Let's assume that we can find this $\underline{v}$, and then prove ourselves wrong. → $[A \cdot \underline{v} = 1 \cdot \underline{v}$ ← for $\lambda = 1$

3) Combining 1) & 2): $\boxed{T(\underline{v}) = A\underline{v} = 1 \cdot \underline{v}}$

for any $\underline{v}$ ↑ ↑ for the $\underline{v}$ in 2)

$\Rightarrow \boxed{T(\underline{v}) = \underline{v}}$ ← Thus, if $\underline{v}$ is the eigenvector of the rotation matrix A (with $\lambda = 1$), then the rotation does not change it!

4) Looking at the figure, we see that such a vector cannot exist: rotation changes <u>any</u> vector!

Thus, our assumption that a $\underline{v}$ with $A\underline{v} = 1 \cdot \underline{v}$ exists, was wrong, $\Rightarrow \lambda = 1$ cannot be an eigenvalue!