Sec. 4.7 - Part 2.

③ Diagonalization of a symmetric matrix

Recall Thm. 19:

$$\begin{pmatrix} n\times n \ A \text{ is diagonalizable,} \\ \text{i.e. } A = VDV^{-1} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} A \text{ has } n \\ \text{lin. indep. eigenvectors} \end{pmatrix}$$

$$[\underline{v}_1, \dots \underline{v}_n] \equiv V$$

Thus, for a generic A, we cannot tell whether it is diagonalizable or not until we actually find all its eigenvalues & eigenvectors and verify that they form a lin. indep. set in $\mathbb{R}^n$. (Thm. 15 says that if A has n distinct λ's, then it does have n lin. indep. eigenvectors, but this still requires finding all eigenvalues...)

However, **when A is symmetric, it is always diagonalizable!** (Note: symmetric $\Rightarrow$ diagonalizable, but a diagonalizable matrix is not necessarily symmetric.)

We will show this in 3 Steps.

Step 1: Define an **orthogonal** matrix and its properties.

Step 2: Show that any A is "triangularizable": $A = QTQ^{-1}$, where Q is orthogonal, T is (upper) triangular.

Step 3: Show that if A is symmetric, then $T = D$, where D is diagonal. Then $A = QDQ^{-1}$, i.e. diagonalizable.

Step 1: Orthogonal matrices.

a Definition & main property

Def: A square matrix Q (with real entries) is orthogonal if it is invertible, and $Q^{-1} = Q^T$. In other words,

$$(Q \text{ is orthogonal}) \iff (Q^TQ = QQ^T = I)$$

(Note: If Q is orthogonal, so is Q^T.)

Interpretation: Rewrite $Q \equiv [q_1, \dots q_n]$. columns

Then

$$Q^T = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \text{ rows}$$

$$Q^TQ = \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \cdot \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ q_2^Tq_1 & q_2^Tq_2 & \cdots & q_2^Tq_n \\ \vdots & \vdots & & \vdots \\ q_n^Tq_1 & q_n^Tq_2 & \cdots & q_n^Tq_n \end{bmatrix}$$

"rows" "columns"

by Def.

$$\overset{\text{by Def.}}{\equiv} I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \Rightarrow$$

$q_1^Tq_1 = q_2^Tq_2 = \dots q_n^Tq_n = 1$, $q_i^T q_j = 0$ for all $i \neq j$.

Thus, the set $\{q_1, \dots q_n\}$ is orthonormal! (sec. 3.7)

$$\boxed{(Q \text{ is orthogonal}) \iff (\text{Columns (and rows) of } Q \text{ are all orthonormal})}$$

MUST READ Ex.5 in textbook.

b Orthogonal matrices & orthogonal transformations

Claim: An orthogonal matrix is the matrix of an orthogonal lin. transformation (Sec. 3.7).

Recall that orthogonal lin. transformations preserve angles between any vectors and lengths of all vectors.
So we'll need to show that orthogonal matrices do the same.

Thm, 21 Let Q be $n \times n$ orthogonal matrix.

(a) Let $\underline{x}$ be any vector in $\mathbb{R}^n$. Then: $\|Q\underline{x}\| = \|\underline{x}\|$.
(Q preserves length of any $\underline{x}$)

(b) Let $\underline{x}, \underline{y}$ be any two vectors in $\mathbb{R}^n$. Then

$$\underline{x}^T \underline{y} = (Q\underline{x})^T (Q\underline{y})$$

(Q preserves the angle between $\underline{x}$ & $\underline{y}$; see the discussion below)

(c) **$\det Q = (+1)$ or (-1)**

Proof of (a): Need to show: l.h.s. = r.h.s.

r.h.s. $= \|\underline{x}\| = \sqrt{\underline{x}^T \underline{x}}$ (Sec. 1.6)

l.h.s. $= \|Q\underline{x}\| = \sqrt{(Q\underline{x})^T (Q\underline{x})} = \sqrt{(\underline{x}^T Q^T)(Q\underline{x})} = \sqrt{\underline{x}^T (Q^T Q)\underline{x}} \overset{\text{by Def. of } Q}{=}$

$= \sqrt{\underline{x}^T I \underline{x}} = \sqrt{\underline{x}^T \underline{x}} =$ r.h.s. ✓

Proof of (b): similar (at home)

Proof of (c): @ home.

Discussion of (b): Recall that in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$\underline{x}^T \underline{y} = \|\underline{x}\| \cdot \|\underline{y}\| \cdot \cos\alpha$, α = angle between $\underline{x}$ & $\underline{y}$.

So $\boldsymbol{\cos\alpha = \underline{x}^T \underline{y} / (\|\underline{x}\| \cdot \|\underline{y}\|)}$. $(*)$

The angle between vectors in $\mathbb{R}^n$ is defined by the same formula.

In $(*)$: $\left\{\begin{array}{l} \bullet \text{ numerator is preserved by } Q \text{ (by (b))}; \\ \bullet \text{ denominator is preserved by } Q \text{ (by (a))}. \end{array}\right.$

Thus the angle α between $\underline{x}$ & $\underline{y}$ is preserved.

Step 2: Triangularization of an arbitrary A.

Thm. 22 (Schur's thm.)
Let A be $n \times n$ and has only real eigenvalues (see below). Then there is an orthogonal Q s.t.

$$A = QTQ^T \qquad (*)$$

where T is $n \times n$ upper-triangular.

Note 1 If eigenvalues of A are not real, Q will also be complex-valued, but conceptually the theorem will still hold. We won't go into that.

Note 2 Since $Q^T = Q^{-1}$, then $(*)$ means that A is similar to T (hence they have the same eigenvalues). Also,

$$A = QTQ^T \iff T = Q^T A Q \qquad (**)$$

Note 3 Thm. 22 allows one to show why the geom. multiplicity $\leq$ alg. multiplicity (See p. 21-20 - optional)

Proof for 2×2 matrix

1) A 2×2 matrix A has at least one eigenvector corresponding to the eigenvalue λ:

$$A\underline{u} = \lambda \underline{u}.$$

We can always normalize $\underline{u}$ s.t.

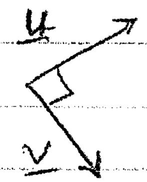
$$\|\underline{u}\| = \sqrt{\underline{u}^T\underline{u}} = 1.$$

2) Consider a vector $\underline{v}$ s.t.:

$\underline{v} \perp \underline{u}$ and

$\|\underline{v}\| = 1$.

$\underline{v}^T \underline{u} = \underline{u}^T \underline{v} = 0$.



Note 1: $\underline{v}$ is not necessarily an eigenvector of A!

Will need in HW problems

Note 2: E.g., if $\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, then $\underline{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$

3) Consider a matrix $Q = [\underline{u}, \underline{v}]$.
Since $\{\underline{u}, \underline{v}\}$ is an orthonormal set, ⇒ Q is orthogonal.
Now consider

$A\underline{u} = \lambda \underline{u}$

$$Q^T A Q = \begin{bmatrix} \underline{u}^T \\ \underline{v}^T \end{bmatrix} A [\underline{u}, \underline{v}] = \begin{bmatrix} \underline{u}^T \\ \underline{v}^T \end{bmatrix} [\lambda \underline{u}, A\underline{v}]$$

$$= \begin{bmatrix} \underline{u}^T \lambda \underline{u} & \underline{u}^T A \underline{v} \\ \underline{v}^T \lambda \underline{u} & \underline{v}^T A \underline{v} \end{bmatrix}$$

$\underline{v}^T \underline{u} = 0$

$$= \begin{bmatrix} \lambda \cdot 1 & \underline{u}^T A \underline{v} \\ 0 & \underline{v}^T A \underline{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}}_{T}.$$

Thus, $Q^T A Q = T \Rightarrow A = Q T Q^T$,
where Q is orthogonal.

Thm. 22 has thus been proved.

Discuss: What should $\underline{v}^T A \underline{v}$ be? Recall Thm. 14.

Ex. 2 Let $A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$.

(a) Find an orthogonal Q and an upper-Δ T s.t. $A = QTQ^T$ (or $T = Q^T A Q$).

(b) Use this result to compute A^2.

Sol'n: (a)

1) Find eigenvalue of A:

$$\begin{vmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0 \Rightarrow (\lambda-4)^2 = 0$$

$\Rightarrow \lambda = 4$ is a double eigenvalue.

2) Find the eigenvector of A:

$$\begin{pmatrix} 5-4 & -1 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \underset{(a=b)}{a - b = 0} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big/ \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{2}}.$$

3) Construct Q:

$$\underline{v} = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \Rightarrow$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$Q^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

4) Find $T = Q^T A Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5-1 & -5-1 \\ 1+3 & -1+3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -6 \\ 4 & 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 8 & -4 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix}$$

λ

as promised by the general theory!

(b) $A^2 = (QTQ^T)(QTQ^T) = QT(\underbrace{Q^TQ}_{I})TQ^T$

$= QTITQ^T$

$= Q\,T^2Q^T.$

$$T^2 = \begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 16+0 & -8-8 \\ 0+0 & 0+16 \end{pmatrix} = \begin{pmatrix} 16 & -16 \\ 0 & 16 \end{pmatrix}.$$

Note: $\boxed{T^2 \neq \begin{pmatrix} 4^2 & 2^2 \\ 0^2 & 4^2 \end{pmatrix}}$ $\boxed{!!!}$

See the note on p. 21-6, where we said that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^k \neq \begin{pmatrix} a^k & b^k \\ c^k & d^k \end{pmatrix}$

(Only for diagonal matrices does one have
$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^k = \begin{pmatrix} d_1^k & 0 \\ 0 & d_2^k \end{pmatrix}.)$$

Continuing our calculation,

$$A^2 = Q\,T^2 Q^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & -16 \\ 0 & 16 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ -16 & 16 \end{pmatrix} = \begin{pmatrix} 24 & -8 \\ 8 & 8 \end{pmatrix}.$$

Step 3 = Thm. 23

Let A be $n \times n$, real & symmetric matrix. Then there is an orthogonal Q s.t.
$$A = QDQ^T,$$
where D is diagonal.

Proof: For any $A = Q\overset{\text{upper-}\Delta}{T}Q^T$, $\Rightarrow T = Q^TAQ$.

A is symmetric $\Rightarrow$ $A^T = A$.

Then consider:

$$T^T = (Q^T A Q)^T \overset{(AB)^T = B^T A^T}{=} Q^T A^T Q \overset{A^T = A}{=} Q^T A Q = T.$$

$\uparrow$ lower-Δ $\qquad\qquad$ $\uparrow$ upper-Δ

Therefore, T must be diagonal: $T = D$.

Thus $\quad A = QDQ^T$,

q.e.d.

!! For the final exam (and beyond), remember: a symmetric matrix is always diagonalizable.

④ Additional properties of symmetric matrices.

Property 1 If A is real & symmetric and λ_1, λ_2 are its distinct eigenvalues, then the corresponding eigenvectors are orthogonal.

Proof:

Have: $\quad A = A^T$,

$$A\underline{v}_1 = \lambda_1 \underline{v}_1, \quad A\underline{v}_2 = \lambda_2 \underline{v}_2, \quad \lambda_1 \neq \lambda_2$$

Want: $\quad \underline{v}_1^T \underline{v}_2 = 0$.

Consider $\quad \underbrace{\underline{v}_2^T A \underline{v}_1}_{\text{scalar}} = \underline{v}_2^T (A\underline{v}_1) = \underline{v}_2^T \lambda_1 \underline{v}_1 = \lambda_1 \underline{v}_2^T \underline{v}_1$.

scalar, = its transpose; so

$$\underline{v}_2^T A \underline{v}_1 = (\underline{v}_2^T A \underline{v}_1)^T = \underline{v}_1^T A^T \underline{v}_2 \underset{A^T = A}{=} \underline{v}_1^T A \underline{v}_2 \underset{\text{similar to above}}{=} \lambda_2 \underline{v}_1^T \underline{v}_2$$

Thus:

$$\lambda_1 \underline{v}_2^T \underline{v}_1 = \lambda_2 (\underline{v}_1^T \underline{v}_2) \overset{\text{scalar = its transponse}}{=} \lambda_2 (\underline{v}_2^T \underline{v}_1).$$

Can only be true for $\lambda_1 \neq \lambda_2$ if $\boxed{\underline{v}_2^T \underline{v}_1 = 0.}$

q.e.d.

Corollary: If A is real & symmetric, then it is possible to choose eigenvectors of A so that they would form an orthogonal basis in $\mathbb{R}^n$.

Idea of proof:

2) The eigenvectors corresponding to distinct λ's are orthogonal by Property 1).

1) A is real & symmetric and hence is diagonalizable (Thm. 23). Hence it has n lin. indep. eigenvectors (Thm. 19).

3) The eigenvectors corresponding to repeated eigenvalues can be made orthogonal by Gram-Schmidt orthogonalization (Sec. 3.6). (Technical details omitted.)

Property 2 If A is real and symmetric, then its eigenvalues are always real.

Proof (skip; available upon request)

1) If $\underline{u}$ is an eigenvector of A, then so is $\underline{u}^*$:

$(A\underline{u} = \lambda \underline{u})^* \Rightarrow A \overset{A \text{ is real}}{\leftarrow} \underline{u}^* = \lambda^* \underline{u}^*$

2) Consider $a \equiv (\underline{u}^*)^T A \underline{u} = \underline{u}^* \lambda \underline{u} = \lambda \underbrace{(\underline{u}^*)^T \underline{u}}_{\text{always real}}$

On the other hand, (A is real, $A^T = A$)

$(a^*)^T = (((\underline{u}^*)^T A \underline{u})^*)^T = (\underline{u}^T A \underline{u}^*)^T = (\underline{u}^*)^T A \underline{u} = a.$

Thus, $a = a^*$, $\Rightarrow$ a is real. Then

$\lambda = a \leftarrow \text{real} / ((\underline{u}^*)^T \underline{u}) \leftarrow \text{real} = \text{real}.$

5 Spectral decomposition of a real symmetric matrix

Q: If we know A, we can compute its eigenvalues and eigenvectors.
Is the converse true?
I.e., if we know the eigenvalues & eigenvectors of a matrix, can we reconstruct the matrix?

A: Yes, if A is real & symmetric (in some* other cases, too, but that is technically more complex).

Property 3 Let A be $n\times n$, real, & symmetric, and let $\{\lambda_1, \dots \lambda_n\}$, $\{\underline{u}_1, \dots \underline{u}_n\}$ be its eigenvalues with corresponding eigenvectors. Then

$$\boxed{A = \lambda_1 \underbrace{\underline{u}_1 \underline{u}_1^T}_{n\times n \text{ matrix}} + \lambda_2 \underline{u}_2 \underline{u}_2^T + \dots + \lambda_n \underline{u}_n \underline{u}_n^T} \quad (\bigstar)$$

spectral decomposition of A.

Proof: 1) By Prop. 1 (Corollary), $\{\underline{u}_1, \dots \underline{u}_n\}$ form an orthonormal basis in $\mathbb{R}^n$.

2) Then any $\underline{x}$ in $\mathbb{R}^n$ can be written as

$$\underline{x} = c_1 \underline{u}_1 + \dots + c_n \underline{u}_n,$$

where (Sec. 3.6) $c_k = (\underline{u}_k^T \underline{x}) / \underbrace{(\underline{u}_k^T \underline{u}_k)}_{\|\underline{u}_k\|^2 = 1} = \underline{u}_k^T \underline{x}$.

3) Let B be the r.h.s. of $(\bigstar)$. Compare $A\underline{x}$ and $B\underline{x}$.

$$A\underline{x} = A(c_1 \underline{u}_1 + \dots + c_n \underline{u}_n) = c_1 \lambda_1 \underline{u}_1 + \dots + c_n \lambda_n \underline{u}_n. \quad (\blacksquare)$$

$$B\underline{x} = (\lambda_1 \underline{u}_1 \underline{u}_1^T + \lambda_2 \underline{u}_2 \underline{u}_2^T + \ldots)\underline{x}$$

$$= \lambda_1 \underline{u}_1 (\underline{u}_1^T \underline{x}) + \lambda_2 \underline{u}_2 (\underline{u}_2^T \underline{x}) + \ldots$$

$$= \lambda_1 \underline{u}_1 c_1 + \lambda_2 \underline{u}_2 c_2 + \ldots \lambda_n \underline{u}_n c_n .$$

This is the same as $(\blacksquare)$, $\Rightarrow$ $A\underline{x} = B\underline{x}$ for any $\underline{x}$, $\Rightarrow$ $A = B$.

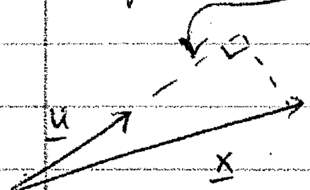
q.e.d.

Interpretation (for 2×2)

See Ex. 3 on p. 21-20

$$\boxed{A = \lambda_1 \underline{u}_1 \underline{u}_1^T + \lambda_2 \underline{u}_2 \underline{u}_2^T .} \qquad (*)$$

Sec. 3.7: If $\|\underline{u}\| = 1$, then $\underline{u}\,\underline{u}^T = P_{\underline{u}}$ = projection matrix on $\underline{u}$.

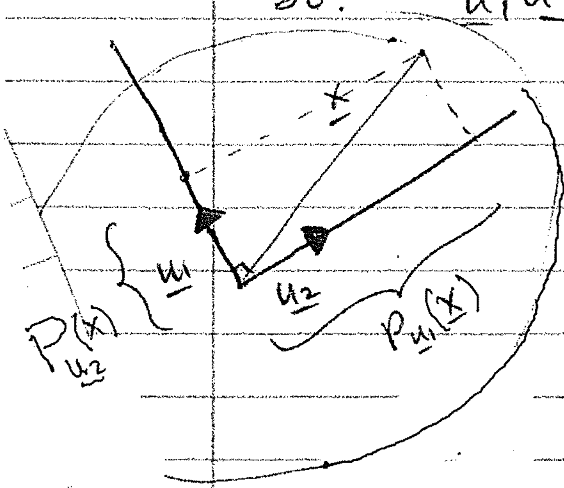


$$P_{\underline{u}}\underline{x} = (\underline{u}\,\underline{u}^T)\underline{x} = \underline{u} \cdot (\underline{u}^T\underline{x})$$

$\underline{u}$: vector directed along $\underline{u}$. $(\underline{u}^T\underline{x})$: coordinate of $\underline{x}$ along $\underline{u}$.

So: $\underline{u}_1 \underline{u}_1^T = P_{\underline{u}_1}$, $\underline{u}_2 \underline{u}_2^T = P_{\underline{u}_2}$

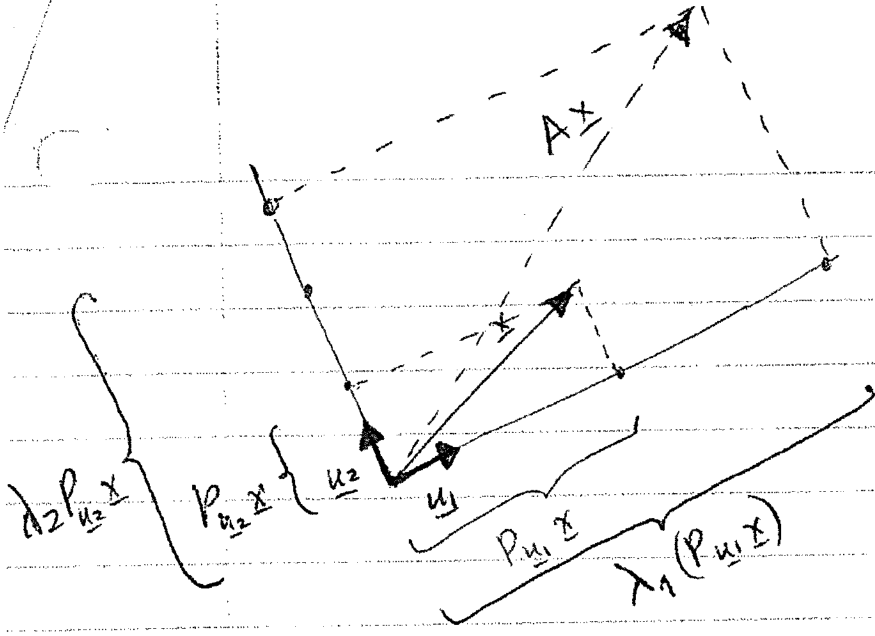
Then $(*)$ says:



$$\boxed{A = \lambda_1 P_{\underline{u}_1} + \lambda_2 P_{\underline{u}_2}}$$

$$A\underline{x} = \underbrace{\lambda_1 \underbrace{(P_{\underline{u}_1}\underline{x})}_{\text{project } \underline{x} \text{ on } \underline{u}_1}}_{\text{stretch by } \lambda_1} + \underbrace{\lambda_2 \underbrace{(P_{\underline{u}_2}\underline{x})}_{\text{project } \underline{x} \text{ on } \underline{u}_2}}_{\text{stretch by } \lambda_2}$$

↓ see next page



Let $\lambda_1 = 2$
$\lambda_2 = 3$

If we compute $A^2\underline{x}$, $A^3\underline{x}$, ... $A^{25}\underline{x}$, ...,

the resulting vector will tend to align with $\underline{u}_2$.

<u>Conclusion</u>: A symmetric matrix is the sum of projections on its eigenvectors, accompanied by stretches by the corresponding eigenvalue!

(If A is not symmetric but is diagonalizable, the situation is more complicated, but similar.)

<u>HW</u> (Part 2): <u>Sec. 4.7</u> 13, 14 [yes], 15, 17;
28 [proceed as in proof for (a)],
29, 30 [Use definition: Compute Q^TQ and $(AB)^T(AB)$.];
33, 35, 36 [In addition, calculate A^5, by the same method as in Ex. 1-5.]

<u>Ex. 4</u> (sidenote) (for the final) Show that $P_{\underline{u}_1} \cdot P_{\underline{u}_2} = \mathcal{O}$ (zero matrix)

$(\underline{u}_1\underline{u}_1^T)(\underline{u}_2\underline{u}_2^T) = \underline{u}_1(\cancel{\underline{u}_1^T\underline{u}_2})^{0}\underline{u}_2^T = 0 \cdot \underbrace{\underline{u}_1\underline{u}_2^T}_{\text{some matrix}} = \mathcal{O}$.

+ <u>Sec. 4.5</u> : #23
25, 26, 27 (b,c).

<u>Ex. 3</u> Given (*), compute A^2.

$$A^2 = (\lambda_1 \underline{u}_1 \underline{u}_1^T + \lambda_2 \underline{u}_2 \underline{u}_2^T)(\lambda_1 \underline{u}_1 \underline{u}_1^T + \lambda_2 \underline{u}_2 \underline{u}_2^T)$$

$$= \lambda_1^2 \underline{u}_1 \cancelto{1}{(\underline{u}_1^T \underline{u}_1)} \underline{u}_1^T + \lambda_1 \lambda_2 \underline{u}_1 \cancelto{0}{(\underline{u}_1^T \underline{u}_2)} \underline{u}_2^T +$$
$$\lambda_2 \lambda_1 \underline{u}_2 \cancelto{0}{(\underline{u}_2^T \underline{u}_1)} \underline{u}_1^T + \lambda_2^2 \underline{u}_2 \cancelto{1}{(\underline{u}_2^T \underline{u}_2)} \underline{u}_2^T$$

$$= \lambda_1^2 \underline{u}_1 \underline{u}_1^T + \lambda_2^2 \underline{u}_2 \underline{u}_2^T .$$

So what is A^k ?

Optional Let's show <u>why a simple (= non-repeated) eigenvalue of A can have only one eigenvector</u> (and not more).
By Thm. 22, A and $T = Q^T A Q$ have the same eigenvalues (because $Q^T = Q^{-1}$), so it suffices to prove the statement for T. Since T is upper-Δ, its eigenvalues = its diagonal entries (Thm. 14). So if λ is an eigenvalue, then
$$T = \begin{pmatrix} d_1 & * & & \\ 0 & \ddots & & * \\ & & \lambda & \\ & 0 & & \ddots \; d_n \end{pmatrix}.$$

Consider $T\underline{x} = \lambda \underline{x}$, $\Rightarrow (T - \lambda I)\underline{x} = \underline{\theta} \Rightarrow$

$$\left(\begin{array}{ccccc|c} d_1-\lambda & * & * & & & 0 \\ & \ddots \; d_{k-1}-\lambda & * & * & & \vdots \\ & & 0 & d_{k+1}-\lambda & & \vdots \\ & 0 & & & \ddots \; d_n-\lambda & 0 \end{array}\right)$$

$d_k \to \lambda-\lambda$

and all $d_j - \lambda \neq 0$ for $j \neq k$.
This is almost an echelon form, from which we see that there is only one free variable in the solution $\underline{x}$, namely, x_k.
Hence there is only one vector $\underline{x}$ in the null space of $(T-\lambda I)$.