

**Reminder:** *You MUST show your work to get credit.*

## Project 5:

### Visualizing eigenvalues and eigenvectors in $\mathbb{R}^2$

#### Goal

Develop an understanding of the geometrical meaning of eigenvectors and eigenvalues of  $2 \times 2$  matrices.

#### General requirements

- You may work alone or with *one* other person. If you work with someone else, hand in *one* answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups. **Please read “My policies on Projects” posted on the course website.**
- Write your answers on the answer sheet provided provided in the last few pages of this document. Staple<sup>1</sup> all the paper showing your *neatly presented*<sup>2</sup> work to the answer sheet.

#### Introduction

Eigenvalues and eigenvectors arise in a great variety of problems in science and engineering. In particular, when matrices are used to describe a given system, the eigenvalues and eigenvectors determine the *long-term behavior* of that system. One example was worked out in the Lecture for Sec. 4.1, where a certain eigenvector predicted an equilibrium distribution of subscribers and non-subscribers to a local newspaper in a town. Another example where you should have observed a similar behavior was in Exercise 2 of Project 1, where the numbers of donating and non-donating alumni reached constant values. Curiously, the key idea behind an early version<sup>3</sup> of Google’s data sorting algorithm is also based on analyzing a distribution of components of a certain eigenvector. There are literally thousands of diverse manifestations of eigenvalues and eigenvalues in Nature and even in our everyday lives.

Given the paramount importance of the eigenvalues and eigenvectors, most software packages have built-in capabilities to compute them. It should be noted that the algorithms used to find the eigenvectors and eigenvalues do **not** use determinants and characteristic polynomials; see # 31 for Sec. 4.2 as to why. The practical algorithms for finding eigenvalues and eigenvectors are studied in courses on Numerical Analysis.

In this project, you will study eigenvalues and eigenvectors of  $2 \times 2$  matrices that represent linear transformations in  $\mathbb{R}^2$ . There are **two main intended outcomes** of this study.

- You will discover that a composition of stretching/shrinking and rotations can sometimes amount to merely stretching or shrinking along certain directions, determined by eigenvectors. A prominent example of this situation occurs in Mechanics of materials, where a combination of pure stresses and shear can be viewed as stresses (or a stress and a strain) along so called principal axes.
- You will then (re)discover that repeated application of the same transformation “favors” directions along eigenvectors corresponding to certain eigenvalues, while “downplaying” directions along other eigenvectors. This is the situation that has been mentioned above in connection with Project 1 and Sec. 4.1.

This project makes heavy use of Matlab. You should use the Matlab resources posted on the course webpage. You may also use other resources or Matlab’s online help.

<sup>1</sup>Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.

<sup>2</sup>I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.

<sup>3</sup>and possibly more recent versions, too

## Exercise 1

In this Exercise, you will find a matrix that describes the consecutive action of two linear transformations.

(a) Copy the code which drew your initial letter in Project 4 into a new file. Create a matrix with the letter's  $x$ - and  $y$ -coordinates and name it something like `original_letter`.

(b) Write a code which applies to this letter two transformations, in the specified order.

- The first transformation stretches the letter horizontally by the factor of 2 and shrinks it vertically by the same factor.
- The second transformation rotates all points in  $\mathbb{R}^2$  by  $\pi/6$  radians about the origin.

*Note 1:* You have not previously derived a matrix of a stretching or shrinking transformation, but this is easy to do. First, from the description, write down what the transformation does to each component of  $\underline{x} = [x_1, x_2]^T$ . Then proceed as you did in Exercise 1 of Project 1. You should obtain a simple, diagonal matrix.

*Note 2:* The matrix of rotation was derived by you in Exercise 1 of Project 1; see also p. 236 of the textbook.

*Note 3:* Matlab programming of applying a transformation to the letter matrix should mimic that in Part 2 of Project 4. See its posted solution if you could not do that part yourself.

(c) Write down a single matrix which represents the *composition* of the above two transformations in the specified order.

*Note 4:* This material is found in the posted Notes for Sec. 3.7.

State whether your composition transformation is linear; briefly give a reason. In the following Exercises, I will refer to this transformation as  $T_C$  and to its matrix as  $C$  (for Composition).

## Exercise 2

Transformation  $T_C$  is a *composition* of stretching/shrinking along two orthogonal directions (horizontal and vertical) *and* a rotation. In this Exercise, you will demonstrate that it is equivalent to *just stretching and shrinking*, but along two non-orthogonal directions. These directions will turn out to be those of the eigenvectors of the matrix  $C$ .

### Preamble

To motivate your work in this Exercise, let us recall that in Sec. 4.7 we showed that diagonalization of a matrix,  $C = VDV^{-1}$  in this case, can be written in an equivalent form:

$$CV = VD, \quad V \equiv [\underline{v}_1, \underline{v}_2], \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \quad (1a)$$

where  $\underline{v}_1, \underline{v}_2$  are linearly independent eigenvectors of  $C$ :

$$C\underline{v}_1 = \lambda_1\underline{v}_1, \quad C\underline{v}_2 = \lambda_2\underline{v}_2. \quad (1b)$$

Let us note that (1b) is consistent with (1a) also in the following sense:

$$(1b) \iff [C\underline{v}_1, C\underline{v}_2] = [\lambda_1\underline{v}_1, \lambda_2\underline{v}_2] \iff C[\underline{v}_1, \underline{v}_2] = [\lambda_1\underline{v}_1 + 0\underline{v}_2, 0\underline{v}_1 + \lambda_2\underline{v}_2]$$
$$\stackrel{\text{Key formula}}{\iff} C[\underline{v}_1, \underline{v}_2] = [\underline{v}_1, \underline{v}_2] \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \iff (1a). \quad (1c)$$

Let us now recall how (1a) can be used to interpret the transformation of an arbitrary vector  $\underline{u}$  in  $\mathbb{R}^2$  by matrix  $C$ . Since  $\underline{v}_1, \underline{v}_2$  are linearly independent, they form a basis in  $\mathbb{R}^2$ , and one can expand  $\underline{u}$  over that basis:

$$\underline{\mathbf{u}} = \underline{\mathbf{v}}_1 c_1 + \underline{\mathbf{v}}_2 c_2 = V \underline{\mathbf{c}}; \quad \underline{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \quad (2)$$

here  $c_1, c_2$  are the coordinates of  $\underline{\mathbf{u}}$  in the basis  $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2\}$ . The transformation of  $\underline{\mathbf{u}}$  by matrix  $C$  can be viewed in that basis as follows:

$$C \underline{\mathbf{u}} \stackrel{\text{Step 1}}{=} C V \underline{\mathbf{c}} \stackrel{\text{Step 2}}{=} V D \underline{\mathbf{c}} \stackrel{\text{Step 3}}{=} [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} \stackrel{\text{Step 4}}{=} \underline{\mathbf{v}}_1 \lambda_1 c_1 + \underline{\mathbf{v}}_2 \lambda_2 c_2. \quad (3)$$

In Step 1, we have written  $\underline{\mathbf{u}}$  in terms of its coordinates in the eigenbasis (i.e., “own” basis) of  $C$ ; see (2).

In Step 2, we have used how  $C$  transforms its “own” basis; see (1a) and (1c).

In Step 3, the “eigencore” of  $C$ , i.e., the diagonal matrix  $D$ , transforms the coordinates.

Finally, in Step 4, these transformed coordinates are used to represent  $C \underline{\mathbf{u}}$  in the eigenbasis of  $C$ .

### Assignment

(a) Use the command `[V,D] = eig(C)` to find the eigenbasis and the eigenvalues of  $C$ ; type `help eig` for details.

(b) Implement Step 1 of the Preamble. That is, find the coordinates of each point of your letter in the eigenbasis of  $C$  by solving the counterpart of Eq. (2). Note that there,  $\underline{\mathbf{u}}$  will be not a vector but a matrix (i.e., a collection of column vectors) representing your letter; see the Example before Part 2 of Project 4. Correspondingly, instead of a column vector  $\underline{\mathbf{c}}$  you will have a matrix, representing the collection of coordinates in the eigenbasis of  $C$ .

*Note 1:* You can use any of Matlab’s commands for solving a linear system, such as the backslash (`help \`) or `inv`. Note that the more familiar command `rref` is *not* a convenient option in this case.

(c) Implement Step 3 of the Preamble. That is, multiply the coordinates by their respective eigenvalues.

*Note 2:* You do *not* need to extract the eigenvalues from matrix  $D$ , obtained in part (a) of this Exercise; nor do you need to break your matrix of coordinates into its rows. *Hint:* Count the dimensions of these matrices and look at Step 3 in Eq. (3).

(d) Implement Step 4 of the Preamble. That is, write the transformed matrix of your letter as the linear superposition of the eigenvectors of  $C$ . Name the result something like `transformed2_letter`.

*Note 3:* Again, you do *not* need to break either  $V$  into its columns or the matrix of transformed coordinates into its rows. See the *Hint* above.

(e) Finally, compare your `transformed2_letter` with the result of *direct* transformation of `original_letter` (see part (a) of Exercise 1) by matrix  $C$ . For reference purposes, I will refer to that result as `transformedbyC_letter`. Thus, you need to plot `transformed2_letter` and `transformedbyC_letter` in the same plot, but use different line styles<sup>4</sup> (not colors!) for the two letters.

*Note 4:* Since `transformed2_letter` and `transformedbyC_letter` are supposed to be identical, it will be impossible to tell them apart in the same plot. To distinguish them and yet be able to see that they are the same, add a small scalar, say 0.005, to each entry of `transformedbyC_letter`. Here Matlab’s convention that  $A + r$  means adding the same scalar  $r$  to each entry of matrix  $A$  will come very handy.

### Exercise 3

Here you will explore a relation between a *repeated* transformation and eigenvalues and eigenvectors of the corresponding matrix.

(a) Find the result of transforming your original letter 5 times with the composition transformation  $T_C$ , repeating the steps of obtaining `transformedbyC_letter` in part (e) of Exercise 2.

Plot the result. Set the ranges along the horizontal and vertical axes of your plot equal by inserting the command

<sup>4</sup>Search for ‘Line specification in Matlab’ online.

axis ('equal') after the plot, but do *not* impose specific limits on the axes.

“Freeze” the figure window by typing `hold on`; this will be needed to combine this plot with another one in part (b) below. Type `help hold` if you need this command explained.

(b) In this “frozen” figure window, plot the eigenvectors of  $C$ , which you found in Exercise 2(a).

How is the *appearance* of your final letter in part (a) of this Exercise related to the eigenvalue(s) and eigenvector(s) of  $C$ ?

*Note 1:* A convenient command to plot a set of vectors whose starting point is the origin, is `plotv` (rather than just `plot`).<sup>5</sup> Note that, again, you will *not* need to extract the individual eigenvectors from  $V$ .

*Note 2:* As you remember, if  $\underline{v}$  is an eigenvector of  $A$ , so is  $r\underline{v}$  for any scalar  $r$ . Matlab chooses this scalar so that the length of all eigenvectors found by command `eig` is 1. However, to make the plotted eigenvectors commensurate in size with the axes’ ranges in the plot in part (a), multiply the eigenvectors by  $r = 5$  (or something close). And again, you do not need to extract the individual eigenvectors from  $V$ .

(c) Compare your answers in parts (a) and (b) with the result of Exercise 2(d) in Project 1 (not 2(d) in this Project!) and with the derivation in Example 1 in the posted Notes for Sec. 4.1. Based on your analysis of the similarities of these situations, fill in the blanks in the text in the Answer sheet.

## Exercise 4

Suppose that instead of transformation  $T_C$  one applies another transformation,  $T_A$ , which differs from  $T_C$  *only* by the order in which stretching/shrinking and rotation are performed. That is, in this new transformation  $T_A$ , rotation is performed first, and stretching/shrinking second.

Suppose that in Exercise 2(e), you have been asked to transform your letter with  $T_A$  instead of  $T_C$ . Without doing any calculations, predict if one should expect, *on general grounds*, if the results here and in Exercise 2(e) will look different. (You do not need to worry about the addition of the small number 0.005 mentioned in Exercise 2(e).)

*Note 1:* Here “different” means that the dimensions and/or orientation of the transformed letters differ by, say, 5% or more.

*Note 2:* You may, of course, apply  $T_A$  to your letter<sup>6</sup> and use your observations in your answer. However, full credit will be given only if you base your answer on a theoretical fact covered in this course, not on your numerical experimentation. This fact is not mentioned in Chap. 3 of the textbook but *is* mentioned explicitly in the posted Notes for Sec. 3.7.

## Bonus

- Review notes for Secs. 3.6 and 3.7 to recognize the action of  $C^k$  with  $k \geq 5$  as being close to a familiar transformation. Describe that transformation in reasonable detail (but without doing any calculations yet).
- Using the notes for Secs. 3.6 and 3.7, compute the matrix of the transformation that you have identified in part (a). Verify whether it is close to  $C^k$ ,  $k \geq 5$ .
- Now consider repeated action of the pure rotation  $T_R$  with the same rotation matrix  $R$  as in Exercise 1. Make a plot of the result of repeated application of  $T_R$  to your initial. Describe in words how the repeated action of  $T_R$  differs from that of the repeated action of  $T_C$ .

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<sup>5</sup>If you are using the Student Version of Matlab, you will need to install either a Deep Learning or a Neural Networks ‘toolbox’ to be able to use `plotv`. In Windows, go to Matlab’s command window, find the ‘APPS’ tab at the top, and go to ‘Get More Apps’ there. It should be similar to that in a Mac.

<sup>6</sup>Just make sure not to mess up your existing code. Save the code that you plan to submit, then save a copy of it under a new name and make all modifications there, not in the original code. You will not need to submit this new code; it is for your use only.

- (d) What *numerical* characteristic(s) of matrices  $R$  and  $C$  makes (or make) the repeated actions of their respective transformations so different? (This question goes partially beyond the material covered in this course, but you should still be able to answer it — or at least state your answer as a plausible hypothesis — by invoking some simple logic.)
- (e) Now, consider another composition transformation,  $T_B$ , where the first transformation is a lengthening along  $x$  by a factor  $b$  and a shortening by the same factor along  $y$ , and the second transformation is the same rotation,  $T_R$ , as before. (Denote the matrix of transformation  $T_B$  as  $B$ .) One can consider a family of such compositions; then the family member with  $b = 2$  is  $T_C$  and the family member with  $b = 1$  is just  $T_R$ . Make a plot, and then describe the repeated action of the family member with  $b = 1.5$  and compare it with the repeated actions of the family members with  $b = 2$  and  $b = 1$ .
- (f) By experimentation, find the *exact* value of  $b$ ,  $b_{\text{thresh}}$ , which demarcates the two behaviors mentioned in part (c). What seems to be the result of the repeated action of  $T_B$  for  $b = b_{\text{thresh}}$ ?  
*Suggestion:* To find  $b_{\text{thresh}}$ , begin by varying the first decimal place of  $b$ . Once you find where the transition between the two behaviors occurs, begin varying only the second decimal place. Then you may repeat this process for the third decimal place, but by then you may already guess what  $b_{\text{thresh}}$  is.
- (g) If you answered the question in part (d), confirm your empirically found value of  $b_{\text{thresh}}$  by an analytical calculation. Provide adequate explanation for all steps of your work.

**Instructions:**

1. There will be a significant number of plots and printouts (of codes) in this project. In order both to organize your work and to save paper, please put all of your printouts and plots<sup>7</sup> into a Word file and print only that file. If this is not done, I will **reduce your score by 5 points**. Append any handwritten work to it as needed.
2. In addition to the printouts of your codes, you must also e-mail me these codes. These codes should produce the action required in the assignment when I run them. For each code which you submit as *either* a hard or soft copy *but not both*, I will **reduce your score by 3 points**.

The subject line of your e-mail must contain the string MATH\_122. It is case-insensitive; thus, math\_122 or Math\_122 will also work. However, the underscore must be present: e.g., math122 or math 122 will *not* work. Moreover, the subject line must contain your name(s) and indicate the fact that this is a submission of codes for Project 5. I will **reduce your score by up to 4 points** if this is not done.

3. Your codes must be named according to this convention: p5\_YourName(s)\_exExercise#.m. E.g., my own code for Exercise 2 would be named p5\_tlakoba\_ex2.m. For each code whose name does not follow this convention, I will **reduce your score by 2 points**.  
Alternatively, you may submit one code, p5\_YourName(s).m, where codes for the individual exercises are separated by Matlab's %% sectioning command.
4. Finally, a **strong request**: Contrary to what you might have learned in other classes, I ask you **not** to put commands `clc` and `close all` at the beginning of your codes. If I want to clear my command window and close figure windows, I will do so myself, without unsolicited outside "help".

**Name(s):** \_\_\_\_\_

**Exercise 1** (24 points)

(b) Write here the matrices of the required stretching/shrinking and of rotation. Write the work on separate pages (or in the Word file).

$$\begin{matrix} \text{stretching/shrinking} \\ \text{matrix} \end{matrix} = \qquad \qquad \qquad \begin{matrix} \text{rotation} \\ \text{matrix} \end{matrix} =$$

(c) Write the matrix of the composition of these transformations here and attach your work on separate pages:

$$C =$$

Is this composition transformation linear? Please explain.

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<sup>7</sup>You will need to create a .png, .jpeg, or .tiff figure to include into a Word document. The syntax in Matlab is: `print -dpng 'foldername/filename'`; see `help print` for more details.

**Exercise 2** (40 points)

(a) List eigenvalues and eigenvectors of  $C$  here; round them to 3 decimal places. Attach your work.

$$\lambda_1 = \quad \lambda_2 = \quad \underline{\mathbf{v}}_1 = \quad \underline{\mathbf{v}}_2 =$$

(b) – (d) Attach the code. Please include a brief justification for matrix dimensions on separate pages.

(e) Attach the code and a plot of the two transformed letters. Mark (by hand is fine) which letter was obtained by which method.

**Exercise 3** (27 points)

(a) and (b) Attach the code and the plot.

How is the appearance of your final letter in part (a) related to the eigenvalue(s) and eigenvector(s) of  $C$ ? Be as specific about the involved eigenvector(s) and its (their) corresponding eigenvalue(s) as possible.

(c) Read the instructions and fill in the blanks:

Let  $A$  be a matrix with *distinct* eigenvalues and let  $\underline{\mathbf{u}}$  be an arbitrary vector. If one multiplies  $\underline{\mathbf{u}}$  by  $A$  \_\_\_\_\_ times, then the result looks similar to the \_\_\_\_\_ of  $A$  that corresponds to the \_\_\_\_\_ eigenvalue.

**Exercise 4** (9 points)

Should the result look different? Make sure to provide an explanation in the attached Word file or on separate pages.

**Bonus** (points per part: (a)→ 7, (b)→ 9, (c)→ 6, (d)→ 11, (e)→ 7, (f)→ 8, (g)→ 11)

Credit for each part will be given only if the solution is at least 60% correct and is clearly presented.

Attach your work on separate pages or/and in the Word file.