Reminder: *You MUST show your work to get credit.*

Project 5:

Visualizing eigenvalues and eigenvectors in $\mathbb{R}^2$

Goal

Develop an understanding of the geometrical meaning of eigenvectors and eigenvalues of 2×2 matrices.

General requirements

- You may work alone or with *one* other person. If you work with someone else, hand in *one* answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups. **Please read "My policies on Projects" posted on the course website.**
- Write your answers on the answer sheet provided provided in the last few pages of this document. Staple (with a staple, not paper clip)[1] all the paper showing your *neatly presented* [2] work to the answer sheet.

Introduction

Eigenvalues and eigenvectors arise in a great variety of problems in science and engineering. In particular, when matrices are used to describe a given system, the eigenvalues and eigenvectors determine the *long-term behavior* of that system. One example was worked out in the Lecture for Sec. 4.1, where a certain eigenvector predicted an equilibrium distribution of subscribers and non-subscribers to a local newspaper in a town. Another example where you should have observed a similar behavior was in Exercise 2 of Project 1, where the numbers of donating and non-donating alumni reached constant values. Curiously, the key idea behind an early version[3] of Google's data sorting algorithm is also based on analyzing a distribution of components of a certain eigenvector. There are literally thousands of diverse manifestations of eigenvalues and eigenvalues in Nature and even in our everyday lives.

Given the paramount importance of the eigenvalues and eigenvectors, most software packages have built-in capabilities to compute them. It should be noted that the algorithms used to find the eigenvectors and eigenvalues do **not** use determinants and characteristic polynomials; see # 31 for Sec. 4.2 as to why. The practical algorithms for finding eigenvalues and eigenvectors are studied in courses on Numerical Analysis.

In this project, you will study eigenvalues and eigenvectors of 2×2 matrices that represent linear transformations in $\mathbb{R}^2$. There are **two main intended outcomes** of this study.

- You will discover that a composition of stretching/shrinking and rotations can sometimes amount to merely stretching or shrinking along certain directions, determined by eigenvectors. A prominent example of this situation occurs in Mechanics of materials, where a combination of pure stresses and shear can be viewed as stresses (or a stress and a strain) along so called principal axes.

- You will then (re)discover that repeated application of the same transformation "favors" directions along eigenvectors corresponding to certain eigenvalues, while "downplaying" directions along other eigenvectors. This is the situation that has been mentioned above in connection with Project 1 and Sec. 4.1.

This project makes heavy use of Matlab. You should use the Matlab resources posted on the course webpage. You may also use other resources or Matlab's online help.

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[1] Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.

[2] I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.

[3] and possibly more recent versions, too

Exercise 1

In this Exercise, you will find a matrix that describes the consecutive action of two linear transformations.

(a) Copy the code which drew your initial letter in Project 4 into a new file. Create a matrix with the letter's x- and y-coordinates and name it `original_letter`.

(b) Write a code which applies to this letter two transformations (see below), in the specified order.

- The first transformation stretches the letter horizontally by the factor of 2 and shrinks it vertically by the same factor.
- The second transformation rotates all points in $\mathbb{R}^2$ by $\pi/6$ radians about the origin.
- Name this twice-transformed letter `transformedbyC_letter`; this name is explained in part (c) below. Make a plot of the `transformedbyC_letter` and include it in your work.

Note 1: You have not previously derived a matrix of a stretching or shrinking transformation, but this is easy to do. First, from the description, write down what the transformation does to each component of $\mathbf{x} = [x_1, x_2]^T$. Then proceed as you did in Exercise 1 of Project 1. You should obtain a simple, diagonal 2×2 matrix.

Note 2: The matrix of rotation was derived by you in Exercise 1 of Project 1; see also p. 236 of the textbook.

Note 3: Your Matlab code applying a transformation to the letter matrix (i.e., to the `original_letter` defined above) should mimic that in Part 2 of Project 4. See its posted solution if you could not do that part yourself.

(c) Write down a single matrix which represents the *composition* of the above two transformations in the specified order. Present a brief calculation, or give a Matlab command, showing how you have found this matrix.

Note 4: This material is found in the posted Notes for Sec. 3.7.

State whether your composition transformation is linear; briefly give a reason. In the following Exercises, I will refer to this transformation as T_C and to its matrix as C (for $\underline{C}$omposition).

Exercise 2

Transformation T_C is a *composition* of stretching/shrinking along two orthogonal directions (horizontal and vertical) *and* a rotation. In this Exercise, you will demonstrate that it is equivalent to *just stretching and shrinking*, but along two non-orthogonal directions. These directions will turn out to be those of the eigenvectors of the matrix C.

Preamble

To motivate your work in this Exercise, let us recall that in Sec. 4.7 we showed that diagonalization of a matrix, $C = VDV^{-1}$ in this case, can be written in an equivalent form:

$$CV = VD, \qquad V \equiv [\underline{\mathbf{v}}_1, \ \underline{\mathbf{v}}_2], \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \tag{1a}$$

where $\underline{\mathbf{v}}_1$, $\underline{\mathbf{v}}_2$ are linearly independent eigenvectors of C:

$$C\underline{\mathbf{v}}_1 = \lambda_1 \underline{\mathbf{v}}_1, \qquad C\underline{\mathbf{v}}_2 = \lambda_1 \underline{\mathbf{v}}_2. \tag{1b}$$

Let us note that (1b) is consistent with (1a) also in the following sense:

$$(1b) \quad \Longleftrightarrow \quad \left[C\underline{\mathbf{v}}_1, C\underline{\mathbf{v}}_2\right] = [\lambda_1\underline{\mathbf{v}}_1, \lambda_2\underline{\mathbf{v}}_2] \quad \Longleftrightarrow \quad C[\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] = [\lambda_1\underline{\mathbf{v}}_1 + 0\,\underline{\mathbf{v}}_2, \ 0\underline{\mathbf{v}}_1 + \lambda_2\underline{\mathbf{v}}_2]$$

$$\overset{\text{Key formula}}{\Longleftrightarrow} \quad C[\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] = [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2] \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \Longleftrightarrow \quad (1a). \tag{1c}$$

Let us now recall how (1a) can be used to interpret the transformation of an arbitrary vector $\underline{\mathbf{u}}$ in $\mathbb{R}^2$ by matrix C. Since $\underline{\mathbf{v}}_1$, $\underline{\mathbf{v}}_2$ are linearly independent, they form a basis in $\mathbb{R}^2$, and one can expand $\underline{\mathbf{u}}$ over that basis:

$$\underline{\mathbf{u}} = \underline{\mathbf{v}}_1 c_1 + \underline{\mathbf{v}}_2 c_2 = V\underline{\mathbf{c}}; \qquad \underline{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \tag{2}$$

here c_1, c_2 are the coordinates of $\underline{\mathbf{u}}$ in the basis $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2\}$. The transformation of $\underline{\mathbf{u}}$ by matrix C can be viewed in that basis as follows:

$$C\underline{\mathbf{u}} \overset{\text{Step 1}}{=} CV\underline{\mathbf{c}} \overset{\text{Step 2}}{=} VD\underline{\mathbf{c}} \overset{\text{Step 3}}{=} [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2]\begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} \overset{\text{Step 4}}{=} \underline{\mathbf{v}}_1\lambda_1 c_1 + \underline{\mathbf{v}}_2\lambda_2 c_2. \tag{3}$$

In Step 1, we have written $\underline{\mathbf{u}}$ in terms of its coordinates in the eigenbasis (i.e., "own" basis) of C; see (2).
In Step 2, we have used how C transforms its "own" basis; see (1a) and (1c).
In Step 3, the "eigencore" of C, i.e., the diagonal matrix D, transforms the coordinates.
Finally, in Step 4, these transformed coordinates are used to represent $C\underline{\mathbf{u}}$ in the eigenbasis of C.

Assignment

(a) Use the command `[V,D] = eig(C)` to find the eigenbasis and the eigenvalues of C; type `help eig` for details.

(b) Implement Step 1 of the Preamble. That is, find the coordinates of each point of your letter in the eigenbasis of C by solving the counterpart of Eq. (2).

Note 1: Here, $\underline{\mathbf{u}}$ will be not a vector but a matrix (i.e., a collection of column vectors) representing your letter, defined in Exercise 1(a). See the Example before Part 2 of Project 4 about representing the matrix of a letter as a collection of vectors. Thus, instead of one column vector $\underline{\mathbf{c}}$, as in Eq. (2), you will have a matrix, representing *the collection* of coordinates of your letter in the eigenbasis of C.

Note 2: You can use any of Matlab's commands for solving a linear system, such as the backslash (`help \`) or `inv`. Note that the more familiar command `rref` is *not* a convenient option in this case.

(c) Implement Step 3 of the Preamble. That is, multiply the coordinates by their respective eigenvalues.

Note 3: You do *not* need to extract the eigenvalues from matrix D, obtained in part (a) of this Exercise; nor do you need to break your matrix of coordinates into its rows. *Hint*: Count the dimensions of these matrices and look at Step 3 in Eq. (3).

(d) Implement Step 4 of the Preamble. That is, write the transformed matrix of your letter as the linear superposition of the eigenvectors of C. Name the result `transformed2_letter`.

Note 4: Again, you do *not* need to break either V into its columns or the matrix of transformed coordinates into its rows. See the *Hint* above.

(e) Finally, compare your `transformed2_letter` with `transformedbyC_letter`, which was the result of *direct* transformation of `original_letter` by matrix C (see Exercise 1(a)). For that, you need to plot `transformed2_letter` and `transformedbyC_letter` in the same plot, but use different line styles[4] (not colors!) for the two letters.

Note 5: Since `transformed2_letter` and `transformedbyC_letter` are supposed to be identical, it will be impossible to tell them apart in the same plot. To distinguish them and yet be able to see that they are the same, add a small scalar, say 0.005, to each entry of `transformedbyC_letter`. Here Matlab's convention that $A + r$ means adding the same scalar r to each entry of matrix A will come very handy.

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[4]Search for 'Line specification in Matlab' online.

Exercise 3

Here you will explore a relation between a *repeated* transformation and eigenvalues and eigenvectors of the corresponding matrix.

(a) Find the result of transforming your original letter 5 times with the composition transformation T_C, repeating the steps of obtaining `transformedbyC_letter` in part (e) of Exercise 2.
Plot the result. Set the ranges along the horizontal and vertical axes of your plot equal by inserting the command `axis('equal')` after the plot, but do *not* impose specific limits on the axes.
"Freeze" the figure window by typing `hold on`; this will be needed to combine this plot with another one in part (b) below. Type `help hold` if you need this command explained.

(b) In this "frozen" figure window, plot the eigenvectors of C, which you found in Exercise 2(a).

How is the *appearance* of your final letter in part (a) of this Exercise related to the eigenvalue(s) and eigenvector(s) of C?

Note 1: A method to plot a vector (as segment of a line) was shown in Part 3 of Project 4.[5]

Note 2: As you remember, if $\underline{v}$ is an eigenvector of A, so is $r\,\underline{v}$ for any scalar r. Matlab chooses this scalar so that the length of all eigenvectors found by command `eig` is 1. However, to make the plotted eigenvectors commensurate in size with the axes' ranges in the plot in part (a), multiply the eigenevectors by $r = 5$ (or something close). As in Exercise 2(c,d), you do not need to extract the individual eigenvectors from V.

(c) Compare your answers in parts (a) and (b) with the result of Exercise 2(d) in Project **1** (<u>not</u> 2(d) in this Project!) and with the derivation in Example 1 in the posted Notes for Sec. 4.1. Based on your analysis of the similarities of these situations, fill in the blanks in the text in the Answer sheet.

Exercise 4

Suppose that instead of transformation T_C one applies another transformation, T_A, which differs from T_C *only* by the order in which stretching/shrinking and rotation are performed. That is, in this new transformation T_A, rotation is performed first, and stretching/shrinking second.

Suppose that in Exercise 2(e), you have been asked to transform your letter with T_A instead of T_C. Without doing any calculations, predict if one should expect, *on general grounds*, if the results here and in Exercise 2(e) will look different. (You do not need to worry about the addition of the small number 0.005 mentioned in Exercise 2(e).)

Note 1: Here "different" means that the dimensions and/or orientation of the transformed letters differ by, say, 5% or more.

Note 2: You may, of course, apply T_A to your letter[6] and use your observations in your answer. However, full credit will be given only if you base your answer on a theoretical fact covered in this course, not on your numerical experimentation. This fact is not mentioned in Chap. 3 of the textbook but *is* mentioned explicitly in the posted Notes for Sec. 3.7.

Bonus

(a) Review notes for Secs. 3.6 and 3.7 to recognize the action of C^k with $k \geq 5$ as being close to a familiar transformation. Describe that transformation in reasonable detail (but without doing any calculations yet).

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[5]Matlab has other commands for plotting vectors, but those require having special toolboxes.
[6]Just make sure not to mess up your existing code. Save the code that you plan to submit, then save a copy of it under a new name and make all modifications there, not in the original code. You will not need to submit this new code; it is for your use only.

(b) Using the notes for Secs. 3.6 and 3.7, compute the matrix of the transformation that you have identified in part (a). Verify whether it is close to to C^k, $k \geq 5$.

(c) Now consider repeated action of the pure rotation T_R with the same rotation matrix R as in Exercise 1. Make one plot where you show several results of repeated application of T_R to your initial (once, twice, ..., five times, ..., until the pattern becomes obvious). Describe in words how the repeated action of T_R differs from that of the repeated action of T_C.

(d) What *numerical characteristic(s)* of matrices R and C makes (or make) the repeated actions of their respective transformations so different? (This question goes partially beyond the material covered in this course. However, based on what you have learned in this course, you should still be able to answer that question — or at least state your answer as a plausible hypothesis — by invoking some simple logic.)

(e) Now, consider another composition transformation, T_B, where the first transformation is a lengthening along x by a factor b and a shortening by the same factor along y, and the second transformation is the same rotation, T_R, as before. (Denote the matrix of transformation T_B as B.) One can consider a family of such compositions; then the family member with $b = 2$ is T_C and the family member with $b = 1$ is just T_R.

Set the value $b = 1.5$ and have Matlab compute the corresponding $T_B \equiv T_{1.5}$. Similarly to what you did in (c), make a plot showing the repeated action of this T_B on your letter. Describe the observed pattern and compare it with the repeated actions of the "T_B-family" members with $b = 2$ and $b = 1$.

(f) By experimentation, find the *exact* value of b, b_{thresh}, which demarcates the two behaviors mentioned in part (c). (I.e., one behavior is "like T_C" and the other is "like T_R".) What seems to be the result of the repeated action of T_B for $b = b_{\text{thresh}}$?

Suggestion: To find b_{thresh}, begin by varying the first decimal place of b. Once you find where the transition between the two behaviors occurs, begin varying only the second decimal place. Then you may repeat this process for the third decimal place, but by then you may already guess what b_{thresh} is.

(g) If you answered the question in part (d), confirm your empirically found value of b_{thresh} by an analytical calculation. Provide adequate explanation for all steps of your work.

Submission instructions:

1. Please submit as a single pdf file only pages 2 and 3 (i.e., the next two pages) of this Answer Sheet, as well as all supporting handwritten work. Please do *not* include either this page with submission instructions or/and the Project description itself.

2. There will be a significant number of plots and printouts (of codes) in this project. In order both to organize your work and to save paper/file size, please put any of your printouts and plots into a Word file (see below), convert it to a pdf, append any handwritten work to the same pdf file, and submit only that file. If this is not done, I will **reduce your score by 5 points**.

   To include a figure into your main file, I recommend the following steps. Create a .png, .jpeg, or .tiff figure; Insert it into a Word document; Convert the Word document into a pdf. The syntax in Matlab for creating a .png file is: print -dpng 'foldername/filename'; see help print for more details. It is similar for other picture formats. Append any handwritten work to it as needed.

3. In addition to the printouts of your codes included in the submitted file, you must also e-mail me these codes. These codes should produce the action required in the assignment when I run them. For each code which you submit as *either* a hard or soft copy *but not both*, I will **reduce your score by 5 points**.

4. The subject line of your e-mail must contain the string MATH_122. It is case-insensitive; thus, math_122 or Math_122 will also work. However, the underscore must be present: e.g., math122 or math 122 will *not* work. Moreover, the subject line must indicate the fact that this is a submission of codes for Project 5. I will **reduce your score by up to 3 points** if this is not done.

5. Your codes must be named according to this convention: p5_YourName(s)_exExercise#.m. E.g., my own code for Exercise 2 would be named p5_tlakoba_ex2.m.
   (Alternatively, you may submit one code, p5_YourName(s).m, where codes for the individual exercises are separated by Matlab's %% sectioning command.)

   For each code whose name does not follow this convention, I will **reduce your score by 2 points**.

6. Finally, suggestion and a **strong request**.

   Suggestion: Include the command clear all at the beginning of your code for Exercise 1.

   Strong request: Contrary to what you might have learned in other classes, I ask you **not** to put commands clc and close all at the beginning of your codes. If I want to clear my command window and close figure windows, I will do so myself, without unsolicited outside "help".

Answer Sheet is continued on next two pages

Reminder: *You MUST staple pages with your neatly presented work to get full credit.*

Name(s): _____

Exercise 1 (24 points)

  (b) Write here the matrices of the required streching/shrinking and of rotation. Write the work on separate pages (or in the Word file).

stretching/shrinking   rotation
      matrix $=$ matrix $=$

  (c) Write the matrix of the composition of these transformations here and attach your work on separate pages:

$$C \;=\;$$

Is this composition transformation linear? Please explain.

Exercise 2 (40 points)

  (a) List eigenvalues and eigenvectors of C here; round them to 3 decimal places. Attach your work.

$$\lambda_1 = \qquad\qquad \lambda_2 = \qquad\qquad \underline{\mathbf{v}}_1 = \qquad\qquad \underline{\mathbf{v}}_2 =$$

  (b) – (d) Attach the code. Please include a brief justification for matrix dimensions on separate pages.

  (e) Attach the code and a plot of the two transformed letters. Mark (by hand is fine) which letter was obtained by which method.

<u>*Answer Sheet is continued on next page*</u>

Continued from previous page

Exercise 3 (27 points)

    (a) and (b)   Attach the code and the plot.

    How is the appearance of your final letter in part (a) related to the eigenvalue(s) and eigenvector(s) of C?
Be as specific about the involved eigenvector(s) and its (their) corresponding eigenvalue(s) as possible.

    (c) Read the instructions and fill in the blanks:

Let A be a matrix with *distinct* eigenvalues and let $\underline{u}$ be an arbitrary vector. If one multiplies $\underline{u}$ by A

_____ times, then the result looks similar to the _____ of A that corresponds to the

_____ eigenvalue.

Exercise 4 (9 points)

    Should the result look different?   Make sure to provide an explanation in the attached Word file or on
separate pages.

Bonus (points per part: (a)→ 7, (b)→ 9, (c)→ 6, (d)→ 12, (e)→ 7, (f)→ 9, (g)→ 12)

    Credit for each part will be given only if the solution is at least 60% correct and is clearly presented.

    Attach your work on separate pages or/and in the Word file.