

Sec. 4.1 The Eigenvalue Problem for 2×2 matrices

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① Motivation

In the first lecture of this course, we considered an example about subscribers and non-subscribers. We observed, by brute-force calculations, some interesting dynamics of their populations.

Later, in Project 1, you observed a similar dynamics of populations of donating and non-donating alumni.

Here we will revisit the subscribers-non-subscribers example and explain the dynamics observed there.

Ex. 1 In a town, people either subscribe or do not subscribe to a local newspaper.

Every year, the numbers of subscribers (S) and non-subscribers (N) change according to:

$$S_{n+1} = 0.7 S_n + 0.5 N_n$$

$$N_{n+1} = 0.3 S_n + 0.5 N_n,$$

i.e., in matrix form:

$$\begin{pmatrix} S \\ N \end{pmatrix}_{n+1} = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} S \\ N \end{pmatrix}_n \equiv A \cdot \begin{pmatrix} S \\ N \end{pmatrix}_n$$

Find what happens to the numbers of subscribers and non-subscribers in, say, 20 years.

Clarification: In Project 1 you showed that

$$\begin{pmatrix} S \\ N \end{pmatrix}_n = A^n \cdot \begin{pmatrix} S \\ N \end{pmatrix}_0.$$

Here, we want to obtain the answer without computing A^n .

Sol'n:

Assumption 1: Assume that there are two vectors \underline{v}_1 and \underline{v}_2 in \mathbb{R}^2 such that

$$\boxed{A \underline{v}_1 = \lambda_1 \underline{v}_1, \quad A \underline{v}_2 = \lambda_2 \underline{v}_2}$$

↑
(Greek letter "lambda")

This assumption does not follow from anywhere (except from subsequent calculations) and looks very strange, as it says that the effect of multiplying \underline{v}_1 and \underline{v}_2 by matrix A amount to simply multiplying them by respective scalars λ_1 and λ_2 . But, you must have seen that this was indeed possible, when you did #43 in the HW for Sec. 1.6.

Assumption 2: Now assume that the above \underline{v}_1 and \underline{v}_2 form a basis in \mathbb{R}^2 . Then any \underline{x} in \mathbb{R}^2 can be expanded over this basis (Sec. 3.4):

$$\underline{x} = c_1 \underline{v}_1 + c_2 \underline{v}_2 .$$

We will take $\underline{x} = \begin{pmatrix} S \\ N \end{pmatrix}_0 \leftarrow$ initial populations;
then

$$\begin{pmatrix} S \\ N \end{pmatrix}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2 .$$

• Using Assumptions 1 & 2, we can now compute

$$\begin{pmatrix} S \\ N \end{pmatrix}_{20} .$$

We will actually start with $\begin{pmatrix} S \\ N \end{pmatrix}_1$, then $\begin{pmatrix} S \\ N \end{pmatrix}_2$,
and then will notice a pattern.

$$\begin{pmatrix} S \\ N \end{pmatrix}_1 = A \begin{pmatrix} S \\ N \end{pmatrix}_0 = A (c_1 \underline{v}_1 + c_2 \underline{v}_2) = c_1 A \underline{v}_1 + c_2 A \underline{v}_2$$

$$\stackrel{\text{(Step 1)}}{=} c_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \underline{v}_2 .$$

$$\begin{pmatrix} S \\ N \end{pmatrix}_2 = A \begin{pmatrix} S \\ N \end{pmatrix}_1 = A (c_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \underline{v}_2)$$

$$= (c_1 \lambda_1) A \underline{v}_1 + (c_2 \lambda_2) A \underline{v}_2 = c_1 \lambda_1 \lambda_1 \underline{v}_1 + c_2 \lambda_2 \lambda_2 \underline{v}_2$$

$$= c_1 \lambda_1^2 \underline{v}_1 + c_2 \lambda_2^2 \underline{v}_2 .$$

Using this pattern, we see that

$$\begin{pmatrix} S \\ N \end{pmatrix}_{20} = c_1 \lambda_1^{20} \underline{v}_1 + c_2 \lambda_2^{20} \underline{v}_2 .$$

thus, instead of computing A^{20} , we need to
compute the simpler quantities (scalars) $\lambda_1^{20}, \lambda_2^{20}$.

What remains is:

- Justify Assumptions 1 & 2 by direct calculation (which would give us λ_1 & λ_2),
and
- See how the above formula explains the observed fact that the populations of subscribers and non-subscribers stabilize at some constant values.

Example 1 adjourned

② Eigenvalues and eigenvectors of 2×2 matrices

Def ("The Eigenvalue Problem"):

Let A be $n \times n$. Let there exist an $\underline{x} \neq \underline{\theta}$ and a scalar λ (Greek letter "lambda") such that:

$$\boxed{A \underline{x} = \lambda \underline{x}},$$

then λ is called an eigenvalue of A , and the (nonzero!) \underline{x} is called the eigenvector corresponding to λ .

Note: We require that $\underline{x} \neq \underline{\theta}$ because otherwise,

$$A \cdot \underline{\theta} = \underline{\theta} = (\text{any } \#) \cdot \underline{\theta}.$$

We'll now discuss how to find these λ and \underline{x} .

Consider $A\underline{x} = \lambda\underline{x} \Rightarrow A\underline{x} - \lambda\underline{x} = \underline{\theta}$

$\Rightarrow (A - \lambda)\underline{x} = \underline{\theta} \dots \leftarrow \text{NO!}$

Cannot subtract a scalar from a matrix!

Try again:

$A\underline{x} - \lambda\underline{x} = \underline{\theta} \Rightarrow A\underline{x} - \lambda \cdot I \underline{x} = \underline{\theta}$

$\Rightarrow (A - \lambda I)\underline{x} = \underline{\theta}$

this is okay!

\nwarrow
n x n
identity
matrix

Thus, to find eigenvalues and eigenvectors, we need to solve

$(A - \lambda I)\underline{x} = \underline{\theta}$ (★)

This involves 2 steps:

Step 1: Find all λ s.t. $(A - \lambda I)$ is singular.

Step 2: Determine the null space of $(A - \lambda I)$; i.e., for each λ , find \underline{x} s.t. (★) holds.

In Ex. 1, A was 2×2 , so in this section we will carry out steps 1 & 2 for 2×2 matrices only.

Let, in general, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$

$(A - \lambda I) = \text{singular} \iff \left[\begin{array}{l} \text{columns of } A - \lambda I \text{ are} \\ \text{lin. dependent, i.e., } \underline{\text{proportional}} \end{array} \right]$
(for 2×2 only)

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Thus, we require:

$$\frac{a-\lambda}{c} = \frac{b}{d-\lambda} \Rightarrow (a-\lambda)(d-\lambda) - bc = 0.$$

Thus, $\left(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \text{singular} \right) \Leftrightarrow \underbrace{\left((a-\lambda)(d-\lambda) - bc = 0 \right)}_{\text{quadratic eqn. for } \lambda}.$

Ex. 1 (resumed) We had $A = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$

Step 1 Find when $(A-\lambda I) = \text{singular}$.

$$\begin{pmatrix} 0.7-\lambda & 0.5 \\ 0.3 & 0.5-\lambda \end{pmatrix} = \text{singular} \Leftrightarrow (0.7-\lambda)(0.5-\lambda) - 0.3 \cdot 0.5 = 0$$

$$\Rightarrow \lambda^2 - 1.2\lambda + 0.2 = 0 \Rightarrow \lambda_{1,2} = 1, 0.2.$$

(use the quadratic formula if inspection fails)

Step 2 For each λ above, find \underline{x} s.t. $(A-\lambda I)\underline{x} = \underline{0}$.

$\lambda = \lambda_1 = 1$ $\begin{pmatrix} 0.7-1 & 0.5 \\ 0.3 & 0.5-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} -0.3 & 0.5 & 0 \\ 0.3 & -0.5 & 0 \end{array} \right). \text{ REF? No, not for } 2 \times 2.$$

The reason: The λ was found so as to make $(A-\lambda I)$ singular. For 2x2 matrices (and only for them!), this means that their columns (and hence rows! - end of Sec.3.5) must be proportional. Since there are only 2 rows, we can then consider only one of them.

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So, for 2x2 matrices, we don't need to use REF.

We just need to check that the rows are indeed proportional (if not - check your arithmetic!), and then solve only the 1st equation:

$$-0.3x_1 + 0.5x_2 = 0 \Rightarrow x_1 = \left(\frac{5}{3}\right)x_2, x_2 = \text{free}.$$

Thus, if we denote the 1st eigenvector as \underline{v}_1 :

$$(A - \lambda_1 I)\underline{v}_1 = \underline{0} \Rightarrow \boxed{\underline{v}_1 = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} x_2, x_2 = \text{free}}$$

$$\lambda = \lambda_2 = 0.2$$

$$\begin{pmatrix} 0.7 - 0.2 & 0.5 \\ 0.3 & 0.5 - 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.3 \end{pmatrix}$ rows are proportional ✓
 $0.5x_1 + 0.5x_2 = 0 \Rightarrow x_1 = -x_2$
 $x_2 = \text{free}$

$$(A - \lambda_2 I)\underline{v}_2 = \underline{0} \Rightarrow \boxed{\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2, x_2 = \text{free}}$$

Note that " x_2 for \underline{v}_1 " and " x_2 for \underline{v}_2 " are not related to each other, so a more correct way to write the answer:

$$\boxed{\begin{aligned} A\underline{v}_1 &= \lambda_1 \underline{v}_1 \Rightarrow \underline{v}_1 = \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \cdot a, a = \text{free} \\ A\underline{v}_2 &= \lambda_2 \underline{v}_2 \Rightarrow \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot b, b = \text{free} \end{aligned}}$$

Note: An eigenvector is always found up to a scalar factor c .

Indeed, if \underline{v} is an eigenvector, then so is $(c \cdot \underline{v})$ for any $c \neq 0$:

$$(A\underline{v} = \lambda \underline{v}) \Leftrightarrow (cA\underline{v} = c\lambda \underline{v}) \Leftrightarrow (A(c\underline{v}) = \lambda(c\underline{v})) \Rightarrow \begin{matrix} (c\underline{v}) \\ \text{an} \\ \text{eigenvector.} \end{matrix}$$

We can now finish Ex. 1.

As in Assumptions 1 & 2 on pp. 17-1, 2, we have found \underline{v}_1 & \underline{v}_2 such that:

A1: $A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$, and

A2: \underline{v}_1 & \underline{v}_2 form a basis in \mathbb{R}^2 ($\underline{v}_1 \neq \underline{v}_2$).

Then:

$$\begin{pmatrix} S \\ N \end{pmatrix}_0 = c_1 \underline{v}_1 + c_2 \underline{v}_2 \quad \left(\text{E.g., } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{3}{8} \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

Then (p. 17-3):

$$\begin{aligned} \begin{pmatrix} S \\ N \end{pmatrix}_{20} &= c_1 \lambda_1^{20} \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \lambda_2^{20} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = c_1 \cdot 1^{20} \cdot \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \cdot 0.2^{20} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= c_1 \cdot 1 \cdot \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} + c_2 \cdot (\approx 0) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \approx c_1 \begin{pmatrix} 5/3 \\ 1 \end{pmatrix}. \end{aligned}$$

This shows: $S_{20}/N_{20} \approx 5/3$, as found in Ex. 1 of Sec. 1.1!

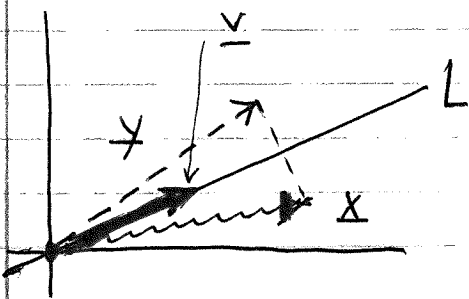
end of Ex. 1

The issues that we have exposed but did not yet address/resolve are:

- (1) Can we be sure that for another A , the eigenvectors \underline{v}_1 & \underline{v}_2 will form a basis?
- (2) How do we generalize this process (= finding of the eigenvalues & eigenvectors) for a $n \times n$ matrix as opposed to just 2×2 ? E.g. for $n=3$?
We'll begin addressing this in Sec. 4.2.

③ Geometric meaning of eigenvectors

Ex. 2(a)



Show that the matrix of reflection about a line in \mathbb{R}^2 always has an eigenvalue $\lambda = 1$.

Sol'n: let A be the matrix of our lin. T .

1) For any vector \underline{x} : $T(\underline{x}) = \underline{y}$ ← some \underline{y} (see picture)

$$\Rightarrow \boxed{A\underline{x} = \underline{y}}$$

2) For the eigenvector \underline{v} : $A\underline{v} = \lambda\underline{v}$ ← for some λ
 $\boxed{A\underline{v} = 1 \cdot \underline{v}}$ ← for $\lambda = 1$

3) Combine 1) & 2): $\boxed{T(\underline{v}) = A\underline{v} = 1 \cdot \underline{v}}$, i.e.
 for any \underline{v} ↑ for this \underline{v}

$$\boxed{T(\underline{v}) = \underline{v}}$$

← thus, if \underline{v} is an eigenvector of the reflection matrix A with $\lambda = 1$, then the reflection does not change \underline{v} !

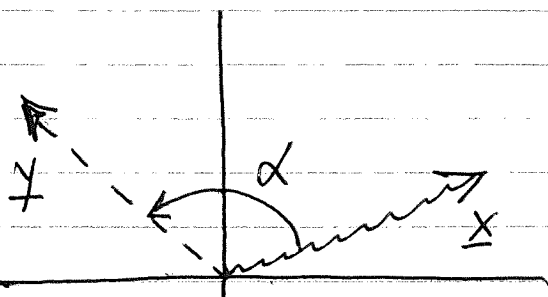
4) Looking at the figure, we see that such a vector indeed exists: it is any vector along the reflection line L .

Thus, we have found a vector \underline{v} that satisfies $T(\underline{v}) = \underline{v}$. Then, tracing back the steps: $T(\underline{v}) = \boxed{A\underline{v} = \underline{v}}$, we see that this \underline{v} is an eigenvector whose eigenvalue $\lambda = 1$.

Ex. 2(b)

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Show that the matrix of a nonzero rotation (by angle $\alpha \neq 0, 360^\circ$, etc.)



does not have an eigenvalue $\lambda = 1$.

Sol'n: We closely follow the steps of Ex. 2(a). Let A be the matrix of rotation.

1) For any vector \underline{x} : $T(\underline{x}) = \underline{y} \leftarrow$ some \underline{y} ; see picture

$$\Rightarrow A \underline{x} = \underline{y}.$$

2) For an eigenvector \underline{v} : $A \underline{v} = \lambda \cdot \underline{v} \leftarrow$ for some λ

let's assume that we can find this \underline{v} , and then prove ourselves wrong. $\rightarrow [A \cdot \underline{v} = 1 \cdot \underline{v} \leftarrow$ for $\lambda = 1$

3) Combining 1) & 2): $\boxed{T(\underline{v}) = A \underline{v} = 1 \cdot \underline{v}}$
for any \underline{v} \leftarrow for the \underline{v} in 2)

$\Rightarrow \boxed{T(\underline{v}) = \underline{v}}$ \leftarrow Thus, if \underline{v} is the eigenvector of the rotation matrix A (with $\lambda = 1$), then the rotation does not change it!

4) Looking at the figure, we see that such a vector cannot exist: rotation changes any vector!

Thus, our assumption that a \underline{v} with $A \underline{v} = 1 \cdot \underline{v}$ exists, was wrong, $\Rightarrow \lambda = 1$ cannot be an eigenvalue!