Project 3:

Concrete application of bases

Goal

Apply the concept of bases in R^n to a practical problem.

General requirements

- You may work alone or with *one* other person. If you work with someone else, hand in *one* answer sheet with both of your names on it.
- No groups bigger than two. No collaboration between groups. **Please read 'My policies on Projects' posted on the course website.**
- Write your answers on the answer sheet provided in the last few pages of this document. Staple (with a staple, not a clip)[1] all the paper showing your *neatly presented* [2] work to the answer sheet.

Introduction

As you know, a basis $\{\mathbf{v}_k\}_{k=1}^n$ in any vector space is such a set of vectors that:
any vector $\mathbf{b}$ in this space can be represented as a linear combination of the basis vectors,

$$\mathbf{b} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n\,, \tag{1}$$

and none of the basis vectors are redundant (i.e., can be replaced by some linear combination of the other basis vectors). The coefficients b_k are called *coordinates* of vector $\mathbf{b}$ in basis $\{\mathbf{v}_k\}_{k=1}^n$. They show "how much" each of the basis vectors "contributes" into $\mathbf{b}$.

An example of a basis that your are well familiar with is the basis of unit coordinate vectors in the plane or in the 3D space. Coordinates in such a basis are the usual vector coordinates you are also well familiar with. Bases naturally appear in many practical engineering applications as well. For example, any sound signal can be represented as a linear superposition of (many) sinusoidal waves with closely spaced frequencies. These sinusoidal waves, called Fourier harmonics, form a basis in the space of all signals with "reasonable" shape. The coordinates of any sound wave in this basis are the strengths of the corresponding Fourier harmonics.

In Exercise 3 of Project 1 you encountered another example of a basis. There, nonfat milk, soy flour, and whey formed a basis of ingredients from which a diet with any[3] desired content of protein, carbohydrates, and fat could be created. The amounts of nonfat milk, soy flour, and whey in any diet can be viewed as "coordinates" of this diet in the basis of these three ingredients. In this Project, you will study in detail a mathematically similar problem occurring in construction business rather than in nutritional science.

[1] Your grade will be reduced by 5% if you hand in a pile of non-stapled sheets.

[2] I will reduce your grade by an amount left to my discretion in each particular case if your work is presented in a messy way and I have to waste time deciphering it.

[3] Well, not really *any*. Some restrictions do apply — see Parts (e) and Bonus-(b) of this Project. However, for our immediate purposes, we will still use the word 'any'.

Problem description

Concrete mixes, used, e.g., for constructing sidewalks and building bridges, are composed of five main materials: cement, water, sand, gravel, and fly ash. By varying the percentages of these materials, mixes of concrete can be produced with differing characteristics. For example, the water to cement ratio affects the strength of the final mix, the sand to gravel ratio affects the "workability" of the mix, and the fly ash to cement ratio affects the durability. Different jobs require concrete with different characteristics. Now, preparing concrete "from scratch", i.e. from the aforementioned five "primordial" ingredients, is time-consuming. Therefore, building supply companies tend to prepare, in the morning of each business day, some basic mixes from which they will be able to quickly create custom mixes for their customers during the business hours.

You are hired as a summer intern by a building supply company. On your first day of work, the company's manager informs you that his company currently stocks three basic mixes of concrete with the following characteristics:

Amount of primordial ingredients [lb] per 60 lb of mix

Ingredient	Super-strong (type S)	All-purpose (type A)	Long-life (type L)
cement	20	18	12
water	10	10	10
sand	20	25	15
gravel	10	5	15
fly ash	0	2	8

The manager then asks you to use your knowledge of Linear Algebra to give answers to the following questions withing one week. (The Hints from your Linear Algebra instructor, whom you have contacted for help, are listed after the questions.)

(a) Does one actually need to stock all three of these mixes, or is any one of them redundant?

(b) Can one get all possible custom mixes from the above three basic ones? If 'yes', explain why. If 'no', explain why and describe all possible custom mixes that one *can* get from them.

(c) A customer has requested 3000 lb of a custom mix with the following proportions of cement, water, sand, gravel, and fly ash: 16, 10, 21, 9, 4. Find the amounts of type S, type A, and type L mixes needed to create this custom mix. Is the solution unique?

(d) A new competitor company in the area has announced that it stocks the following four (instead of only three) basic mixes: type SA, which is an equal mixture of types S and A; type SL, which is an equal mixture of types S and L; type AL, which is an equal mixture of types A and L; and a new type N, which has the following proportions of cement, water, sand, gravel, and fly ash: 10, 20, 5, 15, 10. Determine whether this other company's claim about their being able to produce a larger variety of custom mixes is true *in general* (i.e., from the abstract Linear Algebra perspective).

(e) The customer mentioned in Part (c) is a regular customer of the company you are interning for, and he routinely orders large amounts of the custom mix described in Part (c). Therefore, it is very important to determine whether the new company mentioned in Part (d) is a serious competitor for this customer. Your task is to give an answer to this question.

The following questions will not (at least directly) affect the operation of the company, but the manager has been curious about them for a while. So he asked you to answer them for an extra compensation if you have the time.

(Bonus–(a)) Explain why it is always the case that water makes up one sixth the weight of any custom mix, sand and gravel together make up half the weight, and cement and fly ash together make up the remaining third.

(Bonus–(b)) Is it possible to invent a custom mix that satisfies the conditions in Bonus-(a) but cannot be created from the three basic mixes? If 'yes', invent one. If 'no', explain why it is not possible.

Notes

1. You can represent any mix by a vector $[c, w, s, g, f]^T$ (with the obvious meaning of c, w, etc., and with 'T' standing for the transpose). The basic mixes can be represented by vectors $\mathbf{S}$, $\mathbf{A}$, and $\mathbf{L}$, whose components are given in the Table above. Then any custom mix that it is possible to create out of the basic mixes can be written as their linear combination; see Introduction, and you may also recall what you did in Ex. 3 of Project 1.

2. You are *required* to use software (Mathematica acceptable, but Matlab preferred) to transform matrices whose entries are given numbers (as opposed to variables like x) into Reduced Echelon Form. The commands in Matlab and Mathematica are `rref` and `RowReduce`, respectively. You *may* (but do not have to) use software to manipulate matrices with symbolic entries in parts (b) and (d). An example of such a manipulation is found in a Mathematica file posted alongside this Project. It illustrates a certain trick which is described in Appendix to these Instructions.
Whenever you use software for your calculations, remember to attach printouts with both the commands and the answers.

3. In your computer work, you should setup (most logically, in Part (a)) the vectors $\mathbf{S}$, $\mathbf{A}$, $\mathbf{L}$ only *once*. In all subsequent Parts of the Project, you should *not* define these vectors anew, because that would increase the chance of your making a mistake, and a tiny error in their entries may drastically affect some of your answers. Instead, everywhere in the Project you should use the same $\mathbf{S}$, $\mathbf{A}$, $\mathbf{L}$ that you defined in Part (a).

Hints

For Part (a): What is the linear-algebraic term for "redundant"? (Of course, you must explain the logic of your steps.)

For Part (b): There are at least a couple of ways to answer this question.
One way is related to Example 1 of Section 3.3 and Example 2 of Section 3.4 in the textbook; see also my Note for homework problems ## 15, 17, 19 of Section 3.3, posted on the homework webpage. If you proceed in this way, make sure to read Note 2 above.
Another way is related to topic 1 of the lecture notes for Section 3.3, as well as to Example 1 in lecture notes for Section 3.5.

For Part (c): Set up and solve an appropriate linear system of the form of Eq. (1).

For Part (d): This is similar to Parts (a) and (b). If you follow the second of the approaches mentioned in the Hint for (b), recall the definition of dimension of a subspace.
Technical note: When calculating $\mathbf{SA} = (\mathbf{S} + \mathbf{A})/2$ etc., use Matlab (or Mathematica). Do **not** do it by hand! A tiny arithmetic error may mess up your answer in this and/or the next part.

For Part (e): 1. This is similar to Part (c), *and* you also need to use some common sense.
2. *Clarification*: The new company will be a competitor *if they can* make that custom mix. Conversely, they will not be a competitor *if they cannot* make it. You should focus only on this aspect and not any other aspects, which are not directly related to linear algebra.

For Bonus–(a): Base your answer on a calculation outlined below:
Suppose you mix x_1 buckets of type $\mathbf{S}$, x_2 buckets of type $\mathbf{A}$, and x_3 buckets of type $\mathbf{L}$ (each bucket weighs 60 lb). What is the total weight of this mix? What is the total weight of water in it? Extend this to the combinations of sand and gravel and of cement and fly ash.

For Bonus–(b): There are more than one way to approach this problem. One simple way is hinted at below.

Consider a situation where you can vary the amount of one of the ingredients in a mix while satisfying the constraints of Bonus–(a) and maintaining the total mass of the mix at 60 lb. You can notice that you cannot vary this amount arbitrarily; its allowed range falls within certain bounds, dictated by common sense.

Remark

By comparing your answers for Parts (d) and (e) you could have gotten a feeling that the abstract answers provided by Linear Algebra may sometimes be irrelevant to real-life situations. (If you have also answered the Bonus–(b) questions, this feeling might have become even stronger.) That is, suppose there exist restrictions on coordinates which (the restrictions) are not equations but *inequalities*. Then the Linear Algebra treatment, based on solving linear systems of *equations*, may yield a solution that is not *feasible*. This is, of course, true. However, there is an entire scientific field that focuses on obtaining feasible solutions in such situations. It is called *Linear Programming*; it has many prominent applications in economics and other fields.

Appendix

Here we explain how to perform the calculations done in Example 1 of Section 3.3 of the textbook with software. The issue here is that if you try to use Matlab's or Mathematica's commands for the REF transformation to the *symbolic* matrix at the bottom of p. 178, you will obtain the 3×3 identity matrix rather than the symbolic matrix in Eq. (3) at the top of p. 179. The reason is that both Matlab and Mathematica *assume* that the variables $y_1\, y_2,\ y_3$ are such that the $(3, 3)$th entry of the matrix in Eq. (3) is nonzero and hence divide by that number, obtaining 1. This, of course, will not allow you to obtain the answer to Example 1, which is given by Eq. (4) there.

The workaround is to: (i) at the beginning, "add" an extra *numeric* column that is *linearly independent* of the first two columns of the matrix at the bottom of p. 178; (ii) and then at the end, remove that column from the REF'ed matrix. For example:

$$\text{instead of} \qquad \begin{pmatrix} 2 & 0 & y_1 \\ 1 & 1 & y_2 \\ 0 & 2 & y_3 \end{pmatrix} \qquad \text{use} \qquad \begin{pmatrix} 2 & 0 & \mathbf{0} & y_1 \\ 1 & 1 & \mathbf{0} & y_2 \\ 0 & 2 & \mathbf{1} & y_3 \end{pmatrix};$$

here the "green" column is the one that has been added. The REF of the so modified matrix, computed by the symbolic Matlab or Mathematica, is:

$$\begin{pmatrix} 1 & 0 & \mathbf{0} & y_1/2 \\ 0 & 1 & \mathbf{0} & -y_1/2 + y_2 \\ 0 & 0 & \mathbf{1} & y_1 - 2y_2 + y_3 \end{pmatrix}.$$

Now, the "red" column is the one that you have added earlier to trick Mathematica. So, to go back to your original REF'ed matrix, you need to remove that column. You then obtain the same matrix as in Eq. (3) on p. 179 of the textbook:

$$\begin{pmatrix} 1 & 0 & y_1/2 \\ 0 & 1 & -y_1/2 + y_2 \\ 0 & 0 & y_1 - 2y_2 + y_3 \end{pmatrix}.$$

Therefore, the condition that is to be satisfied by the components of the vector $\underline{\mathrm{y}}$ is given by Eq. (4) on p. 179. (If you are unsure why this condition needs to be imposed, re-read the Note for homework problems ## 15, 17, 19 for Section 3.3, found on the Homework webpage.)

You can now use this trick to obtain algebraic description of spans of other sets of vectors. For example, if you expect *two* algebraic conditions that describe your span rather than one condition as above, you "add" *two* new columns to the matrix, which must be linearly independent from the existing numeric columns. Then, the last *two* entries in the right column of the REF of the modified matrix will yield those two algebraic conditions.

Acknowledgement

This Project is based on Section 2.5 of *ATLAST: Computer exercises for Linear Algebra*, S. Leon, E. Herman, and R. Faulkenberry eds., 2nd ed., Pearson Education, Upper Saddle River, 2003.

Reminder: ***You MUST show all relevant details of your work to get credit. (Skipping minor algebraic calculations is allowed.)***

IMPORTANT NOTE: ***On the page below, give only brief answers. Staple to this page the sheets showing complete details and explanation of your work, including computer work printouts.***

Name(s): ____

(a, 19 points) Is any of the three mixes redundant?

(b, 19 points) Yes or No? Why? Refer to the *full* question on page 2.

(c, 21 points) List the required amounts, in pounds, of mixes S, A, L. Is this answer unique? Why?

(d, 19 points) In abstract Linear Algebra terms, can the new company produce a larger variety of custom mixes than your company?

(e, 22 points) Is the new company a serious competitor for the customer in question?

For Bonus questions, credit will be given only if: (i) your solution is more than 50% correct; (ii) the answer is stated coherently; and (iii) all necessary details are presented.

(Bonus–(a), 7 points) Attach additional pages with your work. You *must* base your solution on the Hint from page 4 to get credit. Hand-waving explanations based on a "they have the same proportions" idea will not earn credit.

(Bonus–(b), 9 points) Attach additional pages with your work.

(Extra Credit from p. 5 of Project 1, 9 points) You may first think why that problem is related to this Project. Attach additional pages with your work.