C Curve of Pursuit

An interesting geometric model arises when one tries to determine the path of a pursuer chasing its prey. This path is called a **curve of pursuit.** These problems were analyzed using methods of calculus circa 1730 (more than two centuries after Leonardo da Vinci had considered them). The simplest problem is to find the curve along which a vessel moves in pursuing another vessel that flees along a straight line, assuming the speeds of the two vessels are constant.

Let's assume that vessel A, traveling at a speed α, is pursuing vessel B, which is traveling at a speed β. In addition, assume that vessel A begins (at time $t = 0$) at the origin and pursues vessel B, which begins at the point (1, 0) and travels up the line $x = 1$. After t hours, vessel A is located at the point $P = (x, y)$, and vessel B is located at the point $Q = (1, \beta t)$ (see Figure 3.18). The goal is to describe the locus of points P; that is, to find y as a function of x.

(a) Vessel A is pursuing vessel B, so at time t, vessel A must be heading right at vessel B. That is, the tangent line to the curve of pursuit at P must pass through the point Q (see Figure 3.18). For this to be true, show that

$$\frac{dy}{dx} = \frac{y - \beta t}{x - 1} . \tag{4}$$

(b) We know the speed at which vessel A is traveling, so we know that the distance it travels in time t is αt. This distance is also the length of the pursuit curve from $(0, 0)$ to (x, y). Using the arc length formula from calculus, show that

$$\alpha t = \int_0^x \sqrt{1 + [y'(u)]^2}\, du . \tag{5}$$

Solving for t in equations (4) and (5), conclude that

$$\frac{y - (x - 1)(dy/dx)}{\beta} = \frac{1}{\alpha} \int_0^x \sqrt{1 + [y'(u)]^2}\, du . \tag{6}$$

(c) Differentiating both sides of (6) with respect to x, derive the first-order equation

$$(x - 1)\frac{dw}{dx} = -\frac{\beta}{\alpha}\sqrt{1 + w^2} ,$$

where $w := dy/dx$.

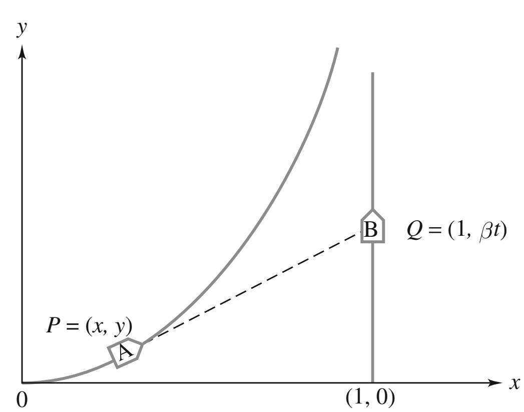

Figure 3.18 The path of vessel A as it pursues vessel B

(d) Using separation of variables and the initial conditions $x = 0$ and $w = dy/dx = 0$ when $t = 0$, show that

$$\frac{dy}{dx} = w = \frac{1}{2}\left[(1 - x)^{-\beta/\alpha} - (1 - x)^{\beta/\alpha}\right] . \tag{7}$$

(e) For $\alpha > \beta$ —that is, the pursuing vessel A travels faster than the pursued vessel B—use equation (7) and the initial conditions $x = 0$ and $y = 0$ when $t = 0$, to derive the curve of pursuit

$$y = \frac{1}{2}\left[\frac{(1 - x)^{1+\beta/\alpha}}{1 + \beta/\alpha} - \frac{(1 - x)^{1-\beta/\alpha}}{1 - \beta/\alpha}\right] + \frac{\alpha\beta}{\alpha^2 - \beta^2} .$$

(f) Find the location where vessel A intercepts vessel B if $\alpha > \beta$.

(g) Show that if $\alpha = \beta$, then the curve of pursuit is given by

$$y = \frac{1}{2}\left\{\frac{1}{2}\left[(1 - x)^2 - 1\right] - \ln(1 - x)\right\} .$$

Will vessel A ever reach vessel B?

D Aircraft Guidance in a Crosswind

Courtesy of T. L. Pearson, Professor of Mathematics, Acadia University (Retired), Nova Scotia, Canada

An aircraft flying under the guidance of a nondirectional beacon (a fixed radio transmitter, abbreviated NDB) moves so that its longitudinal axis always points toward the beacon (see Figure 3.19). A pilot sets out toward an NDB from a point at which the wind is at right angles to the initial direction of the aircraft; the wind maintains this direction. Assume that the wind speed and the speed of the aircraft through the air (its "airspeed") remain constant. (Keep in mind that the latter is different from the aircraft's speed with respect to the ground.)

(a) Locate the flight in the xy-plane, placing the start of the trip at $(2, 0)$ and the destination at $(0, 0)$. Set up the differential equation describing the aircraft's path over the ground. [*Hint:* $dy/dx = (dy/dt)/(dx/dt)$.]

(b) Make an appropriate substitution and solve this equation. [*Hint:* See Section 2.6.]

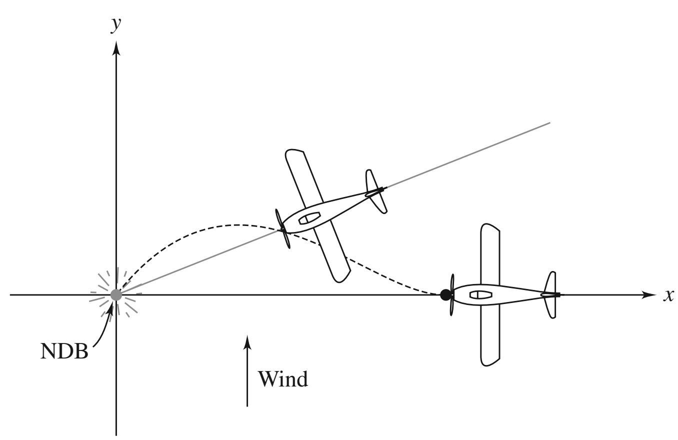

Figure 3.19 Guided aircraft