If the real number β is a root of the equation $f(y) = 0$, then the constant function $y(t) = \beta$ is a solution of $y' = f(y)$. Conversely, if the constant function $y(t) = \beta$ is a solution of $y' = f(y)$, then β must be a root of $f(y) = 0$. Constant solutions of an autonomous differential equation are known as **equilibrium solutions**.

REMARK: It is possible for differential equations that are not autonomous to have constant solutions. For example, $y(t) = 0$ is a solution of $y' = ty + \sin y$ and $y(t) = 1$ is a solution of $y' = (y-1)t^2$. We will refer to any constant solution of a differential equation (autonomous or not) as an equilibrium solution.

EXAMPLE 3

Find the equilibrium solutions (if any) of

$$y' = y^2 - 4y + 3.$$

Solution: The right-hand side of the differential equation is

$$f(y) = y^2 - 4y + 3 = (y-1)(y-3).$$

Therefore, the equilibrium solutions are the constant functions $y(t) = 1$ and $y(t) = 3$. ❖

EXERCISES

Exercises 1–6:

(a) State whether or not the equation is autonomous.

(b) Identify all equilibrium solutions (if any).

(c) Sketch the direction field for the differential equation in the rectangular portion of the ty-plane defined by $-2 \le t \le 2$, $-2 \le y \le 2$.

1. $y' = -y + 1$ **2.** $y' = t - 1$ **3.** $y' = \sin y$

4. $y' = y^2 - y$ **5.** $y' = -1$ **6.** $y' = -ty$

Exercises 7–9:

(a) Determine and sketch the isoclines $f(t, y) = c$ for $c = -1, 0$, and 1.

(b) On each of the isoclines drawn in part (a), add representative direction field filaments.

7. $y' = -y + 1$ **8.** $y' = -y + t$ **9.** $y' = y^2 - t^2$

Exercises 10–13:

Find an autonomous differential equation that possesses the specified properties. [Note: There are many possible solutions for each exercise.]

10. Equilibrium solutions at $y = 0$ and $y = 2$; $y' > 0$ for $0 < y < 2$; $y' < 0$ for $-\infty < y < 0$ and $2 < y < \infty$.

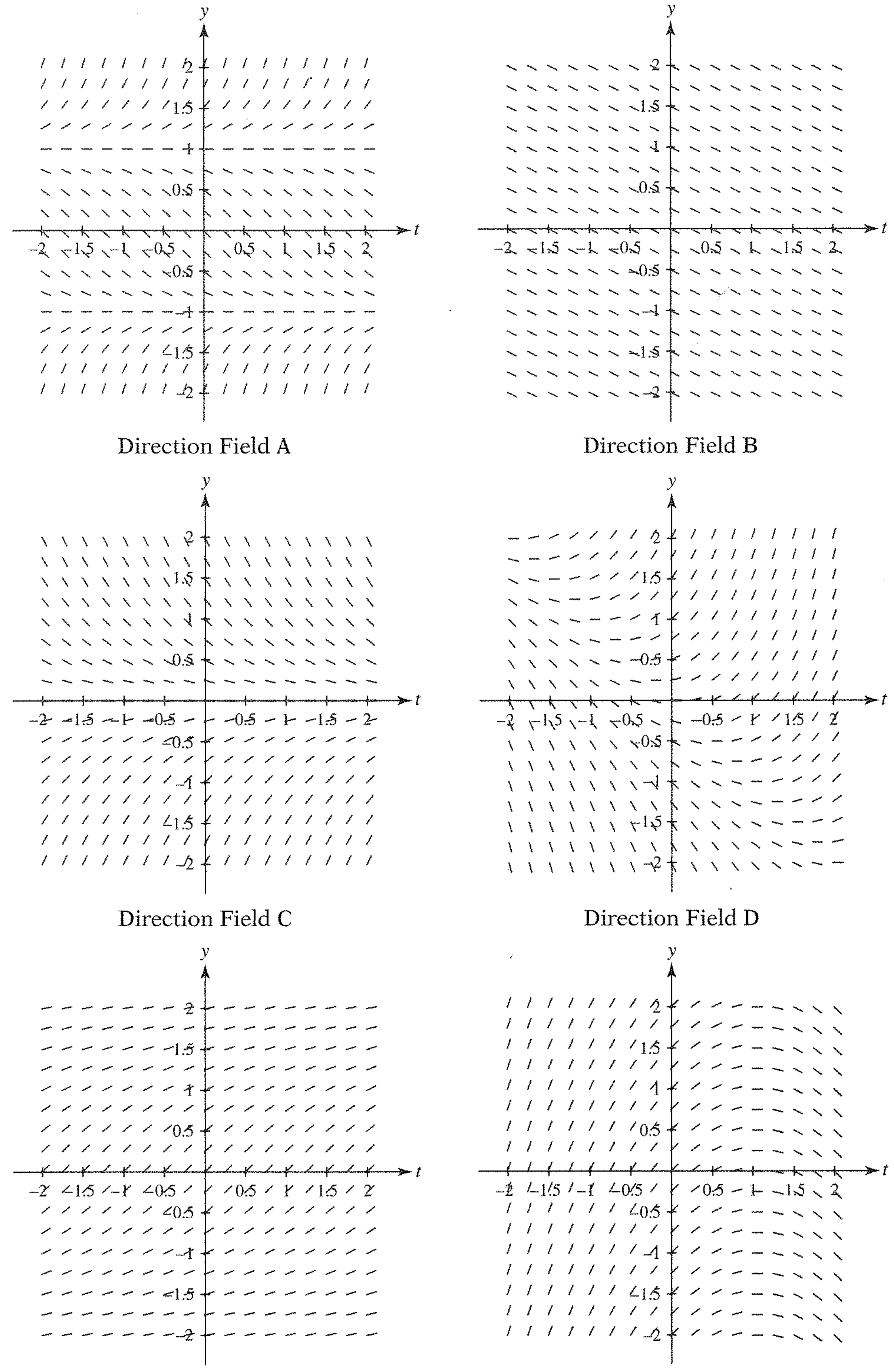
11. An equilibrium solution at $y = 1$; $y' < 0$ for $-\infty < y < 1$ and $1 < y < \infty$.

12. A differential equation with no equilibrium solutions and $y' > 0$ for all y.

13. Equilibrium solutions at $y = n/2$, $n = 0, \pm 1, \pm 2, \ldots$.

Exercises 14–19:

Consider the six direction field plots shown. Associate a direction field with each of the following differential equations.



Direction Field A — Direction Field B — Direction Field C — Direction Field D — Direction Field E — Direction Field F

Figure for Exercises 14–21

14. $y' = -y$ **15.** $y' = -t + 1$ **16.** $y' = y^2 - 1$

17. $y' = -\frac{1}{2}$ **18.** $y' = y + t$ **19.** $y' = \frac{1}{1+y^2}$

20. For each of the six direction fields shown, assume we are interested in the solution that satisfies the initial condition $y(0) = 0$. Use the graphical information contained in the plots to roughly estimate $y(1)$.

21. Repeat Exercise 20 with $y(0) = 0$ as before, but this time estimate $y(-1)$.

CHAPTER 2

First Order Differential Equations

CHAPTER OVERVIEW

2.1 Introduction

First order differential equations arise in modeling a wide variety of physical phenomena. In this chapter we study the differential equations that model applications such as population dynamics, radioactive decay, belt friction, and mixing and cooling.

Chapter 2 has two main parts. The first part, consisting of Sections 2.1–2.4, focuses on first order *linear* differential equations and their applications. The second part, consisting of Sections 2.5–2.9, treats first order *nonlinear* equations. The final section, Section 2.10, introduces numerical techniques, such as Euler's method and Runge-Kutta methods, that can be used to approximate the solution of a first order differential equation.