Theorem 2.1

> Let $p(t)$ and $g(t)$ be continuous functions on the interval (a, b), and let t_0 be in (a, b). Then the initial value problem
>
> $$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$
>
> has a unique solution on the entire interval (a, b).

Notice that the theorem states three conclusions. A solution exists, it is unique, and this unique solution exists on the entire interval (a, b). We will see that determining intervals of existence is considerably more complicated for nonlinear differential equations.

The importance of Theorem 2.1 lies in the fact that it defines the framework within which we can construct solutions. In particular, suppose we are given a linear differential equation $y' + p(t)y = g(t)$ with coefficient functions $p(t)$ and $g(t)$ that are continuous on (a, b). If we impose an initial condition of the form $y(t_0) = y_0$, where $a < t_0 < b$, the theorem tells us there is one and only one solution. Therefore, if we are able to construct a solution by using some technique we have discovered, the theorem guarantees that it is the only solution—there is no other solution we might have overlooked, one obtainable perhaps by a technique other than the one we are using.

EXAMPLE 2

Consider the initial value problem

$$y' + \frac{1}{t(t+2)}y = \frac{1}{t-5}, \qquad y(3) = 1.$$

What is the largest interval (a, b) on which Theorem 2.1 guarantees the existence of a unique solution?

Solution: The coefficient function $p(t) = t^{-1}(t+2)^{-1}$ has discontinuities at $t = 0$ and $t = -2$ but is continuous everywhere else. Similarly, $q(t) = (t-5)^{-1}$ has a discontinuity at $t = 5$ but is continuous for all other values t. Therefore, Theorem 2.1 guarantees that a unique solution exists on each of the following t-intervals:

$$(-\infty, -2), \qquad (-2, 0), \qquad (0, 5), \qquad (5, \infty).$$

Since the initial condition is imposed at $t = 3$, we are guaranteed that a unique solution exists on the interval $0 < t < 5$. (The solution might actually exist over a larger interval, but we cannot ascertain this without actually solving the initial value problem.) ❖

EXERCISES

Exercises 1–10:

Classify each of the following first order differential equations as linear or nonlinear. If the equation is linear, decide whether it is homogeneous or nonhomogeneous.

1. $y' - \sin t = t^2 y$ **2.** $y' - \sin t = ty^2$ **3.** $\dfrac{y'}{y} - y\cos t = t$

4. $y' \sin y = (t^2 + 1)y$

5. $y' \sin t = \dfrac{t^2 + 1}{y}$

6. $2ty + e^t y' = \dfrac{y}{t^2 + 4}$

7. $yy' = t^3 + y \sin 3t$

8. $2ty + e^y y' = \dfrac{y}{t^2 + 4}$

9. $\dfrac{ty'}{(t^4 + 2)y} = \cos t + \dfrac{e^{3t}}{y}$

10. $\dfrac{y'}{(t^2 + 1)y} = \cos t$

Exercises 11–14:

Consider the following first order linear differential equations. For each of the initial conditions, determine the largest interval $a < t < b$ on which Theorem 2.1 guarantees the existence of a unique solution.

11. $y' + \dfrac{t}{t^2 + 1} y = \sin t$

(a) $y(-2) = 1$ (b) $y(0) = \pi$ (c) $y(\pi) = 0$

12. $y' + \dfrac{t}{t^2 - 4} y = 0$

(a) $y(6) = 2$ (b) $y(1) = -1$ (c) $y(0) = 1$ (d) $y(-6) = 2$

13. $y' + \dfrac{t}{t^2 - 4} y = \dfrac{e^t}{t - 3}$

(a) $y(5) = 2$ (b) $y(-\frac{3}{2}) = 1$ (c) $y(0) = 0$
(d) $y(-5) = 4$ (e) $y(\frac{3}{2}) = 3$

14. $y' + (t - 1)y = \dfrac{\ln |t + t^{-1}|}{t - 2}$

(a) $y(3) = 0$ (b) $y\left(\frac{1}{2}\right) = -1$ (c) $y(-\frac{1}{2}) = 1$ (d) $y(-3) = 2$

15. If $y(t) = 3e^{t^2}$ is known to be the solution of the initial value problem

$$y' + p(t)y = 0, \qquad y(0) = y_0,$$

what must the function $p(t)$ and the constant y_0 be?

16. (a) For what value of the constant C and exponent r is $y = Ct^r$ the solution of the initial value problem

$$2ty' - 6y = 0, \qquad y(-2) = 8?$$

(b) Determine the largest interval of the form (a, b) on which Theorem 2.1 guarantees the existence of a unique solution.

(c) What is the actual interval of existence for the solution found in part (a)?

17. If $p(t)$ is any function continuous on an interval of the form $a < t < b$ and if t_0 is any point lying within this interval, what is the unique solution of the initial value problem

$$y' + p(t)y = 0, \qquad y(t_0) = 0$$

on this interval? [Hint: If, by inspection, you can identify one solution of the given initial value problem, then Theorem 2.1 tells you that it must be the only solution.]

(continued)

The graph of $y(t)$ is shown in Figure 2.4(b). Note that $y(t)$ is continuous on the entire t-interval of interest. However, $y(t)$ is not differentiable at $t = 1$.

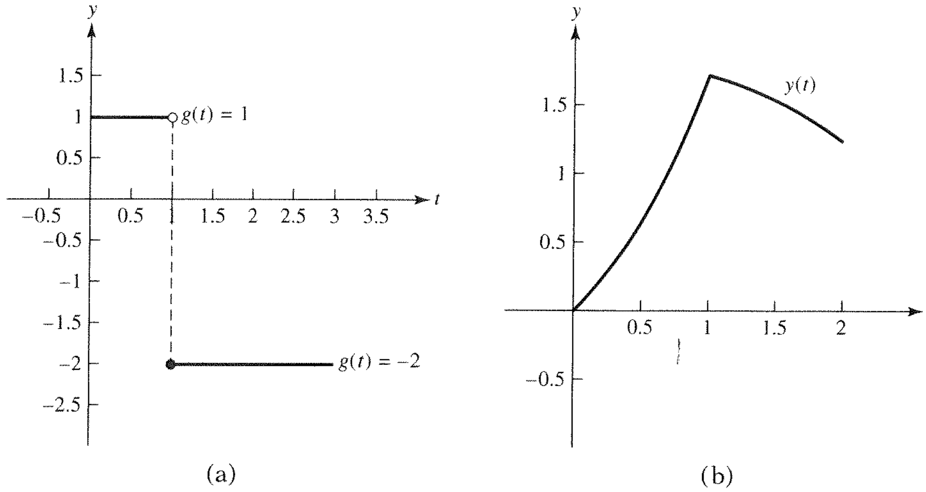


FIGURE 2.4

(a) The coefficient function $g(t)$ of the differential equation $y' - y = g(t)$ in Example 5 has a jump discontinuity at $t = 1$. (b) The solution of $y' - y = g(t)$, $y(0) = 0$ is continuous on the interval $0 \le t \le 2$, but is not differentiable at $t = 1$. ❖

EXERCISES

Exercises 1–10:

For each initial value problem,

(a) Find the general solution of the differential equation.

(b) Impose the initial condition to obtain the solution of the initial value problem.

1. $y' + 3y = 0, \quad y(0) = -3$

2. $2y' - y = 0, \quad y(-1) = 2$

3. $2ty - y' = 0, \quad y(1) = 3$

4. $ty' - 4y = 0, \quad y(1) = 1$

5. $y' - 3y = 6, \quad y(0) = 1$

6. $y' - 2y = e^{3t}, \quad y(0) = 3$

7. $2y' + 3y = e^t, \quad y(0) = 0$

8. $y' + y = 1 + 2e^{-t}\cos 2t, \quad y(\pi/2) = 0$

9. $2y' + (\cos t)y = -3\cos t, \quad y(0) = -4$

10. $y' + 2y = e^{-t} + t + 1, \quad y(-1) = e$

Exercises 11–24:

Find the general solution.

11. $ty' + 4y = 0$

12. $y' + (1 + \sin t)y = 0$

13. $y' - 2(\cos 2t)y = 0$

14. $(t^2 + 1)y' + 2ty = 0$

15. $\dfrac{y'}{(t^2 + 1)y} = 3$

16. $y + e^t y' = 0$

17. $y' + 2y = 1$

18. $y' + 2y = e^{-t}$

19. $y' + 2y = e^{-2t}$

20. $y' + 2ty = t$

21. $ty' + 2y = t^2, \quad t > 0$

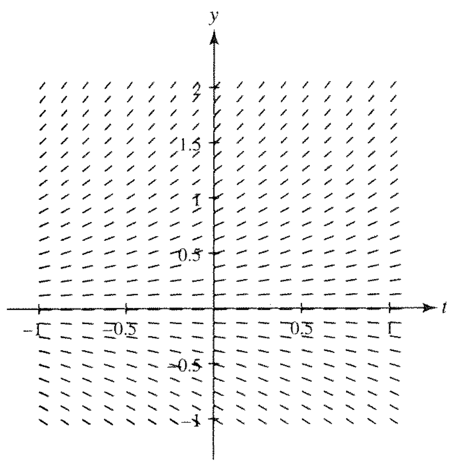
22. $(t^2 + 4)y' + 2ty = t^2(t^2 + 4)$

23. $y' + y = t$

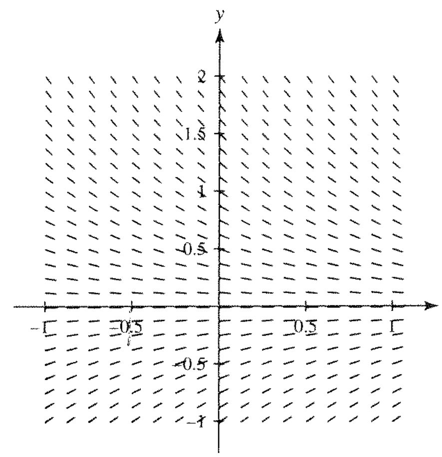
24. $y' + 2y = \cos 3t$

25. Consider the three direction fields shown. Match each of the direction field plots with one of the following differential equations:

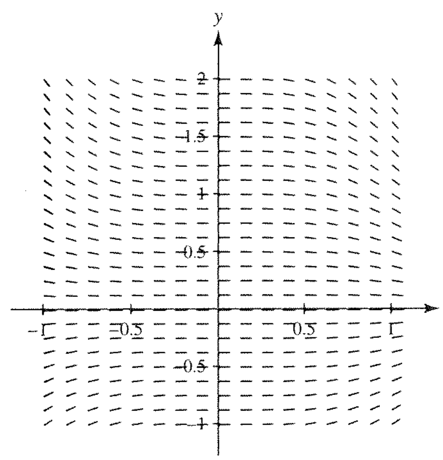
(a) $y' + y = 0$ (b) $y' + t^2 y = 0$ (c) $y' - y = 0$



Direction Field 1



Direction Field 2



Direction Field 3

Figure for Exercise 25

Exercises 26–27:

The graph of the solution of the given initial value problem is known to pass through the (t, y) points listed. Determine the constants α and y_0.

26. $y' + \alpha y = 0, \quad y(0) = y_0$. Solution graph passes through the points (1, 4) and (3, 1).

27. $ty' - \alpha y = 0, \quad y(1) = y_0$. Solution graph passes through the points (2, 1) and (4, 4).

28. Following are four graphs of $y(t)$ versus t, $0 \le t \le 10$, corresponding to solutions of the four differential equations (a)–(d). Match the graphs to the differential equations. For each match, identify the initial condition, $y(0)$.

(a) $2y' + y = 0$ (b) $y' + (\cos 2t)y = 0$

(c) $10y' - (1 - \cos 2t)y = 0$ (d) $10y' - y = 0$

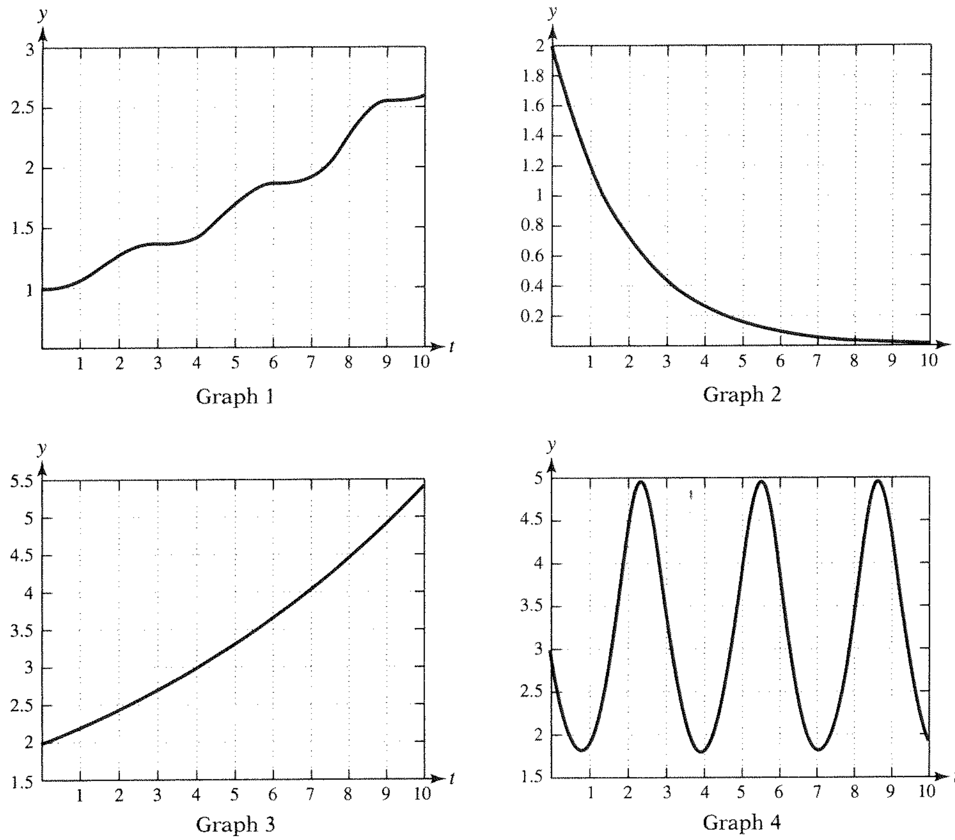


Figure for Exercise 28

29. Antioxidants Active oxygen and free radicals are believed to be exacerbating factors in causing cell injury and aging in living tissue.[1] These molecules also accelerate the deterioration of foods. Researchers are therefore interested in understanding the protective role of natural antioxidants. In the study of one such antioxidant (Hsian-tsao leaf gum), the antioxidation activity of the substance has been found to depend on concentration in the following way:

$$\frac{dA(c)}{dc} = k[A^* - A(c)], \qquad A(0) = 0.$$

In this equation, the dependent variable A is a quantitative measure of antioxidant activity at concentration c. The constant A^* represents a limiting or equilibrium value of this activity, and k is a positive rate constant.

(a) Let $B(c) = A(c) - A^*$ and reformulate the given initial value problem in terms of this new dependent variable, B.

(b) Solve the new initial value problem for $B(c)$ and then determine the quantity of interest, $A(c)$. Does the activity $A(c)$ ever exceed the value A^*?

(c) Determine the concentration at which 95% of the limiting antioxidation activity is achieved. (Your answer is a function of the rate constant k.)

30. The solution of the initial value problem $ty' + 4y = \alpha t^2, y(1) = -\frac{1}{3}$ is known to exist on $-\infty < t < \infty$. What is the constant α?

[1] Lih-Shiuh Lai, Su-Tze Chou, and Wen-Wan Chao, "Studies on the Antioxidative Activities of Hsian-tsao (*Mesona procumbens Hemsl*) Leaf Gum," *J. Agric. Food Chem.*, Vol. 49, 2001, pp. 963–968.

Exercises 31–33:

In each exercise, the general solution of the differential equation $y' + p(t)y = g(t)$ is given, where C is an arbitrary constant. Determine the functions $p(t)$ and $g(t)$.

31. $y(t) = Ce^{-2t} + t + 1$ **32.** $y(t) = Ce^{t^2} + 2$ **33.** $y(t) = Ct^{-1} + 1, \quad t > 0$

Exercises 34–35:

In each exercise, the unique solution of the initial value problem $y' + y = g(t), y(0) = y_0$ is given. Determine the constant y_0 and the function $g(t)$.

34. $y(t) = e^{-t} + t - 1$ **35.** $y(t) = -2e^{-t} + e^t + \sin t$

Exercises 36–37:

In each exercise, discuss the behavior of the solution $y(t)$ as t becomes large. Does $\lim_{t\to\infty} y(t)$ exist? If so, what is the limit?

36. $y' + y + y\cos t = 1 + \cos t, \quad y(0) = 3$

37. $\dfrac{y' - e^{-t} + 2}{y} = -2, \quad y(0) = -2$

38. The solution of the initial value problem $y' + y = e^{-t}, y(0) = y_0$ has a maximum value of $e^{-1} = 0.367\ldots$, attained at $t = 1$. What is the initial condition y_0?

39. Let $y(t)$ be a nonconstant solution of the differential equation $y' + \lambda y = 1$, where λ is a real number. For what values of λ is $\lim_{t\to\infty} y(t)$ finite? What is the limit in this case?

Exercises 40–43:

As in Example 5, find a solution to the initial value problem that is continuous on the given interval $[a, b]$.

40. $y' + \dfrac{1}{t}y = g(t), \quad y(1) = 1; \qquad g(t) = \begin{cases} 3t, & 1 \le t \le 2 \\ 0, & 2 < t \le 3; \end{cases} \qquad [a, b] = [1, 3]$

41. $y' + (\sin t)y = g(t), \quad y(0) = 3; \qquad g(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ -\sin t, & \pi < t \le 2\pi; \end{cases} \qquad [a, b] = [0, 2\pi]$

42. $y' + p(t)y = 2, \quad y(0) = 1; \qquad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \dfrac{1}{t}, & 1 < t \le 2; \end{cases} \qquad [a, b] = [0, 2]$

43. $y' + p(t)y = 0, \quad y(0) = 3; \qquad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\dfrac{1}{t}, & 3 < t \le 4; \end{cases} \qquad [a, b] = [0, 4]$

Exercises 44–45:

In each exercise, you are asked to express the solution in terms of a "special function" [the function $\mathrm{Si}(t)$ in Exercise 44 and $\mathrm{erf}(t)$ in Exercise 45]. Such **special functions** are sufficiently important in applications to warrant giving them names and studying their properties. (A book such as *Handbook of Mathematical Functions* by Abramowitz and Stegun[2] gives the definitions for many important special functions, lists their properties, and has tables of their values. Scientific software such as MATLAB, Mathematica, Maple, and Derive has subroutines for evaluating special functions.)

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[2]Milton Abramowitz and Irene Stegun, *Handbook of Mathematical Functions* (New York: Dover Publications, 1965).

44. Solve $y' - \frac{1}{t}y = \sin t$, $y(1) = 3$. Express your answer in terms of the sine integral, $\mathrm{Si}(t)$, where $\mathrm{Si}(t) = \int_0^t \frac{\sin s}{s}\, ds$. [Note that $\mathrm{Si}(t) = \mathrm{Si}(1) + \int_1^t \frac{\sin s}{s}\, ds$.]

45. Solve $y' - 2ty = 1$, $y(0) = 2$. Express your answer in terms of the error function, $\mathrm{erf}(t)$, where $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\, ds$.

46. Superposition First order linear differential equations possess important superposition properties. Show the following:

(a) If $y_1(t)$ and $y_2(t)$ are any two solutions of the homogeneous equation $y' + p(t)y = 0$ and if c_1 and c_2 are any two constants, then the sum $c_1y_1(t) + c_2y_2(t)$ is also a solution of the homogeneous equation.

(b) If $y_1(t)$ is a solution of the homogeneous equation $y' + p(t)y = 0$ and $y_2(t)$ is a solution of the nonhomogeneous equation $y' + p(t)y = g(t)$ and c is any constant, then the sum $cy_1(t) + y_2(t)$ is also a solution of the nonhomogeneous equation.

(c) If $y_1(t)$ and $y_2(t)$ are any two solutions of the nonhomogeneous equation $y' + p(t)y = g(t)$, then the sum $y_1(t) + y_2(t)$ is *not* a solution of the nonhomogeneous equation.

Exercises 47–48:

Outline of a Proof of Theorem 2.1 The discussion of integrating factors in this section provides a basis for establishing the existence-uniqueness result stated in Theorem 2.1. In particular, consider the initial value problem $y' + p(t)y = g(t)$, $y(t_0) = y_0$, where $p(t)$ and $g(t)$ are continuous on the interval (a, b) and where t_0 is in the interval (a, b). Let $P(t)$ denote the specific antiderivative of $p(t)$ that vanishes at t_0,

$$P(t) = \int_{t_0}^t p(s)\, ds. \tag{11}$$

Since p is continuous on (a, b), it follows from calculus that $P(t)$ is defined and differentiable for all t in (a, b). As an instance of equation (10), define $y(t)$ by

$$y(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^t e^{P(s)} g(s)\, ds. \tag{12}$$

Since g is continuous on (a, b) and $P(t)$ is differentiable on (a, b), it follows from calculus that $G(t) = \int_{t_0}^t e^{P(s)} g(s)\, ds$ is defined and differentiable for all t in (a, b) and that $dG/dt = e^{P(t)} g(t)$.

47. Use the facts above to show that $y(t)$ defined in equation (12) is a solution of the initial value problem $y' + p(t)y = g(t)$, $y(t_0) = y_0$. This explicit construction establishes that at least one solution of the initial value problem exists on the entire interval (a, b).

48. To establish the uniqueness part of Theorem 2.1, assume $y_1(t)$ and $y_2(t)$ are two solutions of the initial value problem $y' + p(t)y = g(t)$, $y(t_0) = y_0$. Define the difference function $w(t) = y_1(t) - y_2(t)$.

(a) Show that $w(t)$ is a solution of the homogeneous linear differential equation $w' + p(t)w = 0$.

(b) Multiply the differential equation $w' + p(t)w = 0$ by the integrating factor $e^{P(t)}$, where $P(t)$ is defined in equation (11), and deduce that $e^{P(t)}w(t) = C$, where C is a constant.

(c) Evaluate the constant C in part (b) and show that $w(t) = 0$ on (a, b). Therefore, $y_1(t) = y_2(t)$ on (a, b), establishing that the solution of the initial value problem is unique.

You can also gain insight into this behavior by examining the direction field for the differential equation, shown in Figure 2.7. Observe from the direction field that $\Theta(t) = S_0$ is an equilibrium solution. This analysis together with Figure 2.7 clearly suggests that the temperature of the body tends toward the temperature of the surroundings as time evolves.

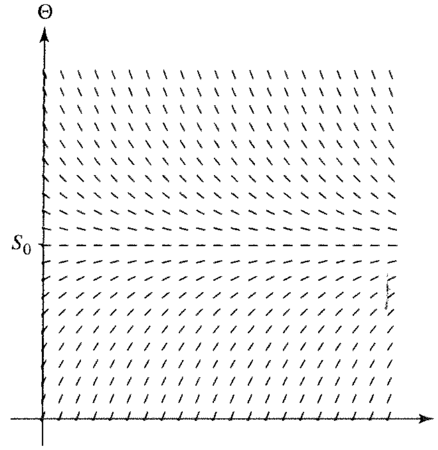

FIGURE 2.7

The direction field for the differential equation $\Theta'(t) = k[S_0 - \Theta(t)]$ that models a cooling problem. The constant function $\Theta(t) = S_0$ is an equilibrium solution. The direction field shows that the temperature of the body, $\Theta(t)$, tends toward the temperature of the surroundings, S_0.

EXERCISES

1. A tank originally contains 100 gal of fresh water. At time $t = 0$, a solution containing 0.2 lb of salt per gallon begins to flow into the tank at a rate of 3 gal/min and the well-stirred mixture flows out of the tank at the same rate.

   (a) How much salt is in the tank after 10 min?

   (b) Does the amount of salt approach a limiting value as time increases? If so, what is this limiting value and what is the limiting concentration?

2. A tank initially holds 500 gal of a brine solution having a concentration of 0.1 lb of salt per gallon. At some instant, fresh water begins to enter the tank at a rate of 10 gal/min and the well-stirred mixture leaves at the same rate. How long will it take before the concentration of salt is reduced to 0.01 lb/gal?

3. An auditorium is 100 m in length, 70 m in width, and 20 m in height. It is ventilated by a system that feeds in fresh air and draws out air at the same rate. Assume that airborne impurities form a well-stirred mixture. The ventilation system is required to reduce air pollutants present at any instant to 1% of their original concentration in 30 min. What inflow (and outflow) rate is required? What fraction of the total auditorium air volume must be vented per minute?

4. A tank originally contains 5 lb of salt dissolved in 200 gal of water. Starting at time $t = 0$, a salt solution containing 0.10 lb of salt per gallon is to be pumped into the tank at a constant rate and the well-stirred mixture is to flow out of the tank at the same rate.

(a) The pumping is to be done so that the tank contains 15 lb of salt after 20 min of pumping. At what rate must the pumping occur in order to achieve this objective?

(b) Suppose the objective is to have 25 lb of salt in the tank after 20 min. Is it possible to achieve this objective? Explain.

5. A 5000-gal aquarium is maintained with a pumping system that passes 100 gal of water per minute through the tank. To treat a certain fish malady, a soluble antibiotic is introduced into the inflow system. Assume that the inflow concentration of medicine is $10te^{-t/50}$ mg/gal, where t is measured in minutes. The well-stirred mixture flows out of the aquarium at the same rate.

(a) Solve for the amount of medicine in the tank as a function of time.

(b) What is the maximum concentration of medicine achieved by this dosing and when does it occur?

(c) For the antibiotic to be effective, its concentration must exceed 100 mg/gal for a minimum of 60 min. Was the dosing effective?

6. A tank initially contains 400 gal of fresh water. At time $t = 0$, a brine solution with a concentration of 0.1 lb of salt per gallon enters the tank at a rate of 1 gal/min and the well-stirred mixture flows out at a rate of 2 gal/min.

(a) How long does it take for the tank to become empty? (This calculation determines the time interval on which our model is valid.)

(b) How much salt is present when the tank contains 100 gal of brine?

(c) What is the maximum amount of salt present in the tank during the time interval found in part (a)? When is this maximum achieved?

7. A tank, having a capacity of 700 gal, initially contains 10 lb of salt dissolved in 100 gal of water. At time $t = 0$, a solution containing 0.5 lb of salt per gallon flows into the tank at a rate of 3 gal/min and the well-stirred mixture flows out of the tank at a rate of 2 gal/min.

(a) How much time will elapse before the tank is filled to capacity?

(b) What is the salt concentration in the tank when it contains 400 gal of solution?

(c) What is the salt concentration at the instant the tank is filled to capacity?

Exercises 8–10:

A tank, containing 1000 gal of liquid, has a brine solution entering at a constant rate of 2 gal/min. The well-stirred solution leaves the tank at the same rate. The concentration within the tank is monitored and is found to be the function of time specified. In each exercise, determine

(a) the amount of salt initially present within the tank.

(b) the inflow concentration $c_i(t)$, where $c_i(t)$ denotes the concentration of salt in the brine solution flowing into the tank.

8. $c(t) = \dfrac{e^{-t/500}}{50}$ lb/gal **9.** $c(t) = \dfrac{1}{20}(1 - e^{-t/500})$ lb/gal **10.** $c(t) = \dfrac{te^{-t/500}}{500}$ lb/gal

11. A 500-gal aquarium is cleansed by the recirculating filter system schematically shown in the figure. Water containing impurities is pumped out at a rate of 15 gal/min, filtered, and returned to the aquarium at the same rate. Assume that passing through the filter reduces the concentration of impurities by a fractional amount α, as shown in the figure. In other words, if the impurity concentration upon entering the filter is $c(t)$, the exit concentration is $\alpha c(t)$, where $0 < \alpha < 1$.

(a) Apply the basic conservation principle (rate of change = rate in − rate out) to obtain a differential equation for the amount of impurities present in the aquarium at time t. Assume that filtering occurs instantaneously. If the outflow concentration at any time is $c(t)$, assume that the inflow concentration at that same instant is $\alpha c(t)$.

(b) What value of filtering constant α will reduce impurity levels to 1% of their original values in a period of 3 hr?

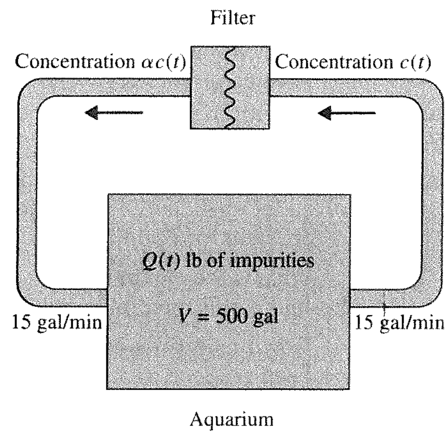

Figure for Exercise 11

12. Consider the mixing process shown in the figure. A mixing chamber initially contains 2 gal of a clear fluid. Clear fluid flows into the chamber at a rate of 10 gal/min. A dye solution having a concentration of 4 oz/gal is injected into the mixing chamber at a rate of r gal/min. When the mixing process is started, the well-stirred mixture is pumped from the chamber at a rate of $10 + r$ gal/min.

(a) Develop a mathematical model for the mixing process.

(b) The objective is to obtain a dye concentration in the outflow mixture of 1 oz/gal. What injection rate r is required to achieve this equilibrium solution? Would this equilibrium value of r be different if the fluid in the chamber at time $t = 0$ contained some dye?

(c) Assume the mixing chamber contains 2 gal of clear fluid at time $t = 0$. How long will it take for the outflow concentration to rise to within 1% of the desired concentration?

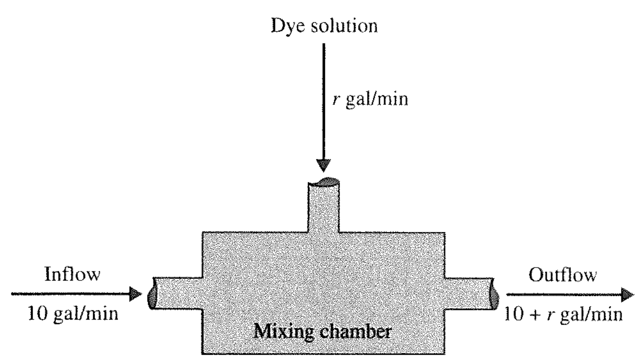

Figure for Exercise 12

13. **Series Connections of Tanks** Consider the sketch shown below, where two ponds are connected and fed by a single stream flowing through them. Pond A holds 500,000 gal of water, while Pond B holds 200,000 gal of water. The fresh water stream flows through these ponds at a rate of 1000 gal/hr. Assume that at some time, say $t = 0$, 1000 lb of a toxin is spilled into Pond A and disperses rapidly enough that a well-stirred assumption is reasonable.

(a) Let $Q_A(t)$ and $Q_B(t)$ denote the amounts of toxin in Ponds A and B, respectively, at time t. Apply the "conservation of salt" principle to each pond and formulate initial value problems governing how the amount of toxin in each pond varies with time.

(b) Solve the two initial value problems for $Q_A(t)$ and $Q_B(t)$. (Because of the way the two ponds are connected by the feeder stream, the problem for Pond A can be solved independently of that for Pond B and the solution, in turn, used to specify the problem for Pond B.)

(c) What is the maximum amount of toxin present in Pond B and at what time after the spill is this maximum value reached?

(d) How much time must elapse before the concentration of toxin in both ponds has been reduced to 1 lb per million gallons?

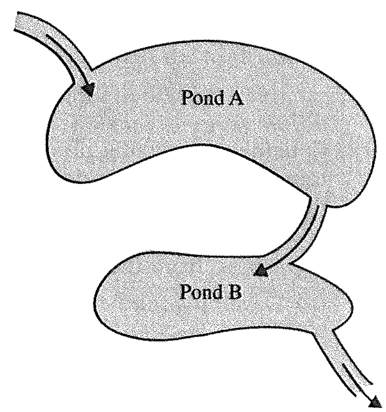

Figure for Exercise 13

14. **Oscillating Flow Rate** A tank initially contains 10 lb of solvent in 200 gal of water. At time $t = 0$, a pulsating or oscillating flow begins. To model this flow, we assume that the input and output flow rates are both equal to $3 + \sin t$ gal/min. Thus, the flow rate oscillates between a maximum of 4 gal/min and a minimum of 2 gal/min; it repeats its pattern every $2\pi \approx 6.28$ min. Assume that the inflow concentration remains constant at 0.5 lb of solvent per gallon.

(a) Does the amount of solution in the tank, V, remain constant or not? Explain.

(b) Let $Q(t)$ denote the amount of solvent (in pounds) in the tank at time t (in minutes). Explain, on the basis of physical reasoning, whether you expect the amount of solvent in the tank to approach an equilibrium value or not. In other words, do you expect $\lim_{t\to\infty} Q(t)$ to exist and, if so, what is this limit?

(c) Formulate the initial value problem to be solved.

(d) Solve the initial value problem. Determine $\lim_{t\to\infty} Q(t)$ if it exists.

15. **Oscillating Inflow Concentration** A tank initially contains 10 lb of salt dissolved in 200 gal of water. Assume that a salt solution flows into the tank at a rate of 3 gal/min and the well-stirred mixture flows out at the same rate. Assume that the inflow

concentration oscillates in time, however, and is given by $c_i(t) = 0.2(1 + \sin t)$ lb of salt per gallon. Thus, as time evolves, the concentration oscillates back and forth between 0 and 0.4 lb of salt per gallon.

(a) Make a conjecture, on the basis of physical reasoning, as to whether or not you expect the amount of salt in the tank to reach a constant equilibrium value as time increases. In other words, will $\lim_{t\to\infty} Q(t)$ exist?

(b) Formulate the corresponding initial value problem.

(c) Solve the initial value problem.

(d) Plot $Q(t)$ versus t. How does the amount of salt in the tank vary as time becomes increasingly large? Is this behavior consistent with your intuition?

Assume Newton's law of cooling applies in Exercises 16–23.

16. A chef removed an apple pie from the oven and allowed it to cool at room temperature (72°F). The pie had a temperature of 350°F when removed from the oven; 10 min later, the pie had cooled to 290°F. How long will it take for the pie to cool to 120°F?

17. The temperature of an object is raised from 70°F to 150°F in 10 min when placed within a 300°F oven. What oven temperature will raise the object's temperature from 70°F to 150°F in 5 min?

18. An object, initially at 150°F, was placed in a constant-temperature bath. After 2 min, the temperature of the object had dropped to 100°F; after 4 min, the object's temperature was observed to be 90°F. What is the temperature of the bath?

Exercises 19–21:

A metal casting is placed in an environment maintained at a constant temperature, S_0. Assume the temperature of the casting varies according to Newton's law of cooling. A thermal probe attached to the casting records the temperature $\theta(t)$ listed. Use this information to determine

(a) the initial temperature of the casting.

(b) the temperature of the surroundings.

19. $\theta(t) = 70 + 270e^{-t}$ °F **20.** $\theta(t) = 390e^{-t/2}$ °F **21.** $\theta(t) = 80 - 40e^{-2t}$ °F

22. Food, initially at a temperature of 40°F, was placed in an oven preheated to 350°F. After 10 min in the oven, the food had warmed to 120°F. After 20 min, the food was removed from the oven and allowed to cool at room temperature (72°F). If the ideal serving temperature of the food is 110°F, when should the food be served?

23. A student performs the following experiment using two identical cups of water. One cup is removed from a refrigerator at 34°F and allowed to warm in its surroundings to room temperature (72°F). A second cup is simultaneously taken from room temperature surroundings and placed in the refrigerator to cool. The time at which each cup of water reached a temperature of 53°F is recorded. Are the two recorded times the same or not? Explain.

2.4 Population Dynamics and Radioactive Decay

In this section, we study simple population models based on first order linear equations. We also examine models for radioactive decay and applications such as radiocarbon dating.

Solution: The general solution of equation (5) is

$$Q(t) = Ce^{-kt},$$

where t is measured in days. Imposing the initial condition, we obtain

$$Q(t) = 50e^{-kt}.$$

As in Example 1, we use the fact that $Q(5) = 43$ mg to determine the decay rate k:

$$k = -\tfrac{1}{5}\ln\tfrac{43}{50} = 0.03016\ldots\ \text{days}^{-1}.$$

After 30 days, therefore, we expect to have

$$Q(30) = 50e^{-k30} = 20.228\ldots\ \text{mg.}$$ ❖

The **half-life** of a radioactive substance is the length of time it takes a given amount of the substance to be reduced to one half of its original amount. Thus, the half-life τ is defined by the equation

$$Q(t+\tau) = \tfrac{1}{2}Q(t).$$

Since $Q(t) = Ce^{-kt}$, this equation reduces to $e^{-k\tau} = 0.5$ and hence

$$\tau = \frac{\ln 2}{k}.$$

For example, the substance in Example 3 has a half-life of about

$$\frac{\ln 2}{0.0302} = 22.95\ \text{days}.$$

If we had 300 mg of the substance at some given time, we would have about 150 mg of the substance 22.95 days later and 75 mg of the substance after 45.9 days.

EXERCISES

Assume the populations in Exercises 1–4 evolve according to the differential equation $P' = kP$.

1. A colony of bacteria initially has 10,000,000 members. After 5 days, the population increases to 11,000,000. Estimate the population after 30 days.
2. How many days will it take the colony in Exercise 1 to double in size?
3. A colony of bacteria is observed to increase in size by 30% over a 2-week period. How long will the colony take to triple its initial size?
4. A colony of bacteria initially has 100,000 members. After 6 days, the population has decreased to 80,000. At that time, 50,000 new organisms are added to replenish its size. How many bacteria will be in the colony after an additional 6 days?
5. Initially, 100 g of a radioactive material is present. After 3 days, only 75 g remains. How much additional time will it take for radioactive decay to reduce the amount present to 30 g?

6. Radioactive decay reduces an initial amount of material by 20% over a period of 90 days. What is the half-life of this material?

7. A radioactive material has a half-life of 2 weeks. After 5 weeks, 20 g of the material is seen to remain. How much material was initially present?

8. After 30 days of radioactive decay, 100 mg of a radioactive substance was observed to remain. After 120 days, only 30 mg of this substance was left.

   (a) How much of the substance was initially present?

   (b) What is the half-life of this radioactive substance?

   (c) How long will it take before only 1% of the original amount remains?

9. Initially, 100 g of material A and 50 g of material B were present. Material A is known to have a half-life of 30 days, while material B has a half-life of 90 days. At some later time it was observed that equal amounts of the two radioactive materials were present. When was this observation made?

10. The evolution of a population with constant migration rate M is described by the initial value problem

$$\frac{dP}{dt} = kP + M, \qquad P(0) = P_0.$$

   (a) Solve this initial value problem; assume k is constant.

   (b) Examine the solution $P(t)$ and determine the relation between the constants k and M that will result in $P(t)$ remaining constant in time and equal to P_0. Explain, on physical grounds, why the two constants k and M must have opposite signs to achieve this constant equilibrium solution for $P(t)$.

11. Assume that the population of fish in an aquaculture farm can be modeled by the differential equation $dP/dt = kP + M(t)$, where k is a positive constant. The manager wants to operate the farm in such a way that the fish population remains constant from year to year. The following two harvesting strategies are under consideration.

   ***Strategy I*:** Harvest the fish at a constant and continuous rate so that the population itself remains constant in time. Therefore, $P(t)$ would be a constant and $M(t)$ would be a negative constant; call it $-M$. (Refer to Exercise 10.)

   ***Strategy II*:** Let the fish population evolve without harvesting throughout the year, and then harvest the excess population at year's end to return the population to its value at the year's beginning.

   (a) Determine the number of fish harvested annually with each of the two strategies. Express your answer in terms of the population at year's beginning; call it P_0. (Assume that the units of k are year^{-1}.)

   (b) Suppose, as in Example 2, that $P_0 = 500{,}000$ fish and $k = 0.0061 \times 52 = 0.3172$ year^{-1}. Assume further that Strategy I, with its steady harvesting and return, provides the farm with a net profit of \$0.75/fish while Strategy II provides a profit of only \$0.60/fish. Which harvesting strategy will ultimately prove more profitable to the farm?

12. Assume that two colonies each have P_0 members at time $t = 0$ and that each evolves with a constant relative birth rate $k = r_b - r_d$. For colony 1, assume that individuals migrate into the colony at a rate of M individuals per unit time. Assume that this immigration occurs for $0 \le t \le 1$ and ceases thereafter. For colony 2, assume that a similar migration pattern occurs but is delayed by one unit of time; that is, individuals migrate at a rate of M individuals per unit time, $1 \le t \le 2$. Suppose we are interested in comparing the evolution of these two populations over the time

interval $0 \le t \le 2$. The initial value problems governing the two populations are

$$\frac{dP_1}{dt} = kP_1 + M_1(t), \qquad P_1(0) = P_0, \qquad M_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 2; \end{cases}$$

$$\frac{dP_2}{dt} = kP_2 + M_2(t), \qquad P_2(0) = P_0, \qquad M_2(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t \le 2. \end{cases}$$

(a) Solve both problems to determine P_1 and P_2 at time $t = 2$.

(b) Show that $P_1(2) - P_2(2) = (M/k)\left(e^k - 1\right)^2$. If $k > 0$, which population is larger at time $t = 2$? What happens if $k < 0$?

(c) Suppose that there is a fixed number of individuals that can be introduced into a population at any time through migration and that the objective is to maximize the population at some fixed future time. Do the calculations performed in this problem suggest a strategy (based on the relative birth rate) for accomplishing this?

13. Radiocarbon Dating Carbon-14 is a radioactive isotope of carbon produced in the upper atmosphere by radiation from the sun. Plants absorb carbon dioxide from the air, and living organisms, in turn, eat the plants. The ratio of normal carbon (carbon-12) to carbon-14 in the air and in living things at any given time is nearly constant. When a living creature dies, however, the carbon-14 begins to decrease as a result of radioactive decay. By comparing the amounts of carbon-14 and carbon-12 present, the amount of carbon-14 that has decayed can therefore be ascertained.

Let $Q(t)$ denote the amount of carbon-14 present at time t after death. If we assume its behavior is modeled by the differential equation $Q'(t) = -kQ(t)$, then $Q(t) = Q(0)e^{-kt}$. Knowing the half-life of carbon-14, we can determine the constant k. Given a specimen to be dated, we can measure its radioactive content and deduce $Q(t)$. Knowing the amount of carbon-12 present enables us to determine $Q(0)$. Therefore, we can use the solution of the differential equation $Q(t) = Q(0)e^{-kt}$ to deduce the age, t, of the radioactive sample.

(a) The half-life of carbon-14 is nominally 5730 years. Suppose remains have been found in which it is estimated that 30% of the original amount of carbon-14 is present. Estimate the age of the remains.

(b) The half-life of carbon-14 is not known precisely. Let us assume that its half-life is 5730 ± 30 years. Determine how this half-life uncertainty affects the age estimate you computed in (a); that is, what is the corresponding uncertainty in the age of the remains?

(c) It is claimed that radiocarbon dating cannot be used to date objects older than about 60,000 years. To appreciate this practical limitation, compute the ratio $Q(60{,}000)/Q(0)$, assuming a half-life of 5730 years.

14. Suppose that 50 mg of a radioactive substance, having a half-life of 3 years, is initially present. More of this material is to be added at a constant rate so that 100 mg of the substance is present at the end of 2 years. At what constant rate must this radioactive material be added?

15. Iodine-131, a fission product created in nuclear reactors and nuclear weapons explosions, has a half-life of 8 days. If 30 micrograms of iodine-131 is detected in a tissue site 3 days after ingestion of the radioactive substance, how much was originally present?

16. U-238, the dominant isotope of natural uranium, has a half-life of roughly 4 billion years. Determine how long it takes for a sample to be reduced in amount by 1% through radioactive decay.