MATH 230

Elementary Differential Equations
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Lecture 1. Introduction

① Motivation

Ex. 1   a) An object starts
at y_0 and moves
along y-axis with
speed v.

Find location $y(t)$ at any t.

Sol'n: $y(t) = y_0 + vt$.

b)   Q: What if $v \neq$ const?   (1)

We know: $\boxed{y'(t) = v(t)}$ ← integrate

$y(t) = \int v(t)\,dt + C$ ②
↳ explicitly account
for arb. constant.

Constant C is found from the initial condition.

However, we now use an integration method which will find C right away. (we'll use the same method many times in this course.)

$$\int_0^t y'(t_1)\,dt_1 = \int_0^t v(t_1)\,dt_1$$

we will explain this notation later

by Fund. Thm. of Calculus $= y(t) - y(0)$.

so:

$$y(t) - y(0) = \int_0^t v(t_1)\,dt_1$$

$$\boxed{y(t) = y_0 + \int_0^t v(t_1)\,dt_1} \qquad (3)$$

Moral: Our goal is:

Given the diff. eqn. $\dfrac{dy}{dt} = f(t,y)$ $\qquad (4a)$

and initial condition $\quad y(t_0) = y_0$ $\qquad (4b)$

We want to find solution $y(t)$.

Note 1 The combination of diff. eqn., (4a), ←(DE)
and initial condition, (4b), is called
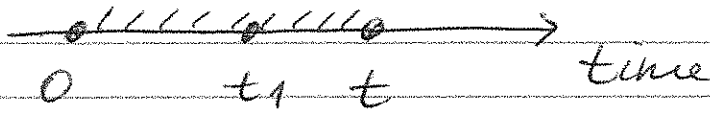initial value problem (IVP).

Note 2 When we solve any DE, we get
an arbitrary constant, c (see (2)).
Its value is found from the initial cond.;
e.g., in (3), $c = y_0$.

Note 3 (about notations)

When we integrate $v(t)$ from 0 to t, we write:

$$\int_0^t v(t_1)\,dt_1$$

this is the time up to which we integrate

This is a "dummy" variable $(0 \leq t_1 \leq t)$.

It must have a different name than the finite integration time.

Sometimes, we may use $\tilde{t}$ (t-tilde) instead of t_1:

$$\int_0^t v(\tilde{t})\,d\tilde{t}.$$

Note 4   DE (4a) is more complicated than DE (1). Indeed, if we write its solution as:

$$y(t) = \int f(t, y(t))\,dt + C, \qquad (5)$$

we see that we cannot integrate because we do not know the unknown solution $y(t)$ inside $f(t, y)$.

So we need to develop methods of how to find $y(t)$ other than by (5).

② Examples of DEs

Ex. 2   Given IVP: const
$$y' = ay \qquad (6)$$
$$y(0) = 3$$

a) Show that the solution of this DE
   is
$$y = Ce^{at} \qquad (C = const)$$

b) Find the solution of the IVP.

Sol'n: a) We are asked to verify
a given solution, so we
simply compare the lhs.
and rhs.

                                        Chain Rule
lhs: $y' = (Ce^{at})' = C(e^{at})' = C \cdot (at)' \cdot e^{at}$

$$= C \cdot a \cdot e^{at}$$

rhs: $a \cdot y = a \cdot Ce^{at}$.

$$lhs = rhs. \checkmark$$

b) $Ce^{at}\big|_{t=0} = 3 \Rightarrow C \cdot e^{0} = 3 \Rightarrow C = 3$.

$$y(t) = 3e^{at}.$$

Note: DE (6) is the most basic DE
which arises in many applications in
all branches of science (physics, chemistry,
...)

It basically says:

$$y' = a \cdot y$$

rate of change of y — is proportional to — instantaneous value of Y.

For example:

sub-Ex. 2(a)

$$\frac{dm}{dt} = -a \cdot m$$

mass of radioactive material
(radioactive decay law :
a given percentage of material decays in
a unit of time).

sub-Ex. 2(B) ← Newton's Cooling Law
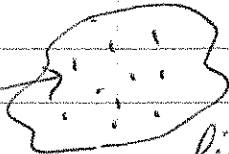
$$\frac{d(T-T_r)}{dt} = -k \cdot (T-T_r)$$

coffee

T_r = room temp.

temperature of coffee

temp. of coffee in excess of room temperature

sub-Ex. 2(c)

$$\frac{dn}{dt} = b \cdot n$$

number of bacteria multiplying in presence of unlimited supply of food.

③ Some terminology

1) $y'(t) = f(t, y(t))$

$t \leftarrow$ independent variable

$y \leftarrow$ dependent variable

2) ordinary DE: $\dfrac{dy}{dt}$ is ordinary derivative.

will not study in math 230 {

Partial DE: $\dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}$

involves partial derivatives of dependent variable $u(x,t)$ w.r.t. two independent variables x, t.

3) Order of DE = order of highest derivative

Ex. 3(a)

$y'' = 1$

$\llcorner$ 2nd der., $\Rightarrow$ order $= 2$

Ex. 3(b)

$\sin(y') + \cos(y''') = 0$

$\llcorner$ 3rd der., $\Rightarrow$ order $= 3$

Ex. 3(c)

$(y'')^5 + (y')^2 + \ln y = 0$

$\llcorner$ 2nd der., $\Rightarrow$ order $= 2$.

4) Autonomous & non-autonomous DE.

$$y' = f(y) \qquad \leftarrow \text{autonomous.}$$

$$y' = y^2 - \sin(e^y).$$
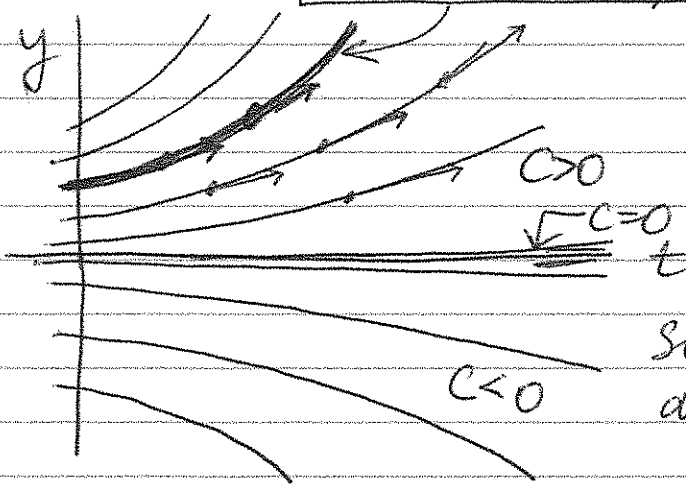
$$y' = f(t,y) \leftarrow \text{non-autonomous}$$

$$y' = y^2 - t$$

④ <u>Direction fields</u>

<u>Idea</u>: Recall Ex. 2.

IVP has <u>1 solution</u> $y(t)$.

But DE has
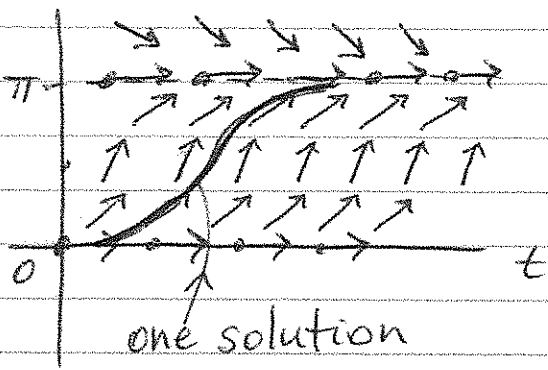∞ many solutions
because of
arb. const C.

So sol'ns of DE
are <u>many curves</u>.

Direction field is the collection of all
these curves with <u>arrows</u> at each point
indicating the slope.

Ex. 4   Sketch the direction field of

$$y' = \sin y$$

one solution

Sol'n:

This is an <u>autonomous</u> DE, $\Rightarrow$ y' is the same along each line $y = $ const.

$y = 0 \Rightarrow \sin y = 0 \Rightarrow y' = 0$

$y = \frac{\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$

$y = \frac{\pi}{2} \Rightarrow \sin y = 1, \qquad y' = 1$

$y = \frac{3\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$

$y = \pi \qquad \sin y = 0, \qquad y' = 0$

$y = \frac{5\pi}{4} \qquad \sin y = -\frac{1}{\sqrt{2}} = -0.7, \quad y' = -0.7$

Equilibrium solution:

A <u>constant</u> solution such that $y' = 0$ for all t:

$$y' = f(t, y_0) = 0.$$

That is, $y(t) = y_0$ is a sol'n for all t.

Ex. 5 (a) In Ex. 4, equilibrium sol'ns
            are $y = 0, \pi, 2\pi$, etc.

Ex. 5 (b) In
$$y' = t(y+1)^2,$$
the equilibrium sol'n is $y = -1$.

Ex. 5 (c) In
$$y' = ty + 1,$$
there are __no__ equilibrium solutions,
because $y' = 0$ implies $y = -1/t$, which
contradicts $y' = 0$ $((-1/t)' = 1/t^2 \neq 0)$.

Ex. 5 (d) In
$$y' = y^2 + 1$$
there are also __no__ equilib. solutions
because $(y^2 + 1) > 0$ for all y.

HW : Sec. 1.2 ## 5, 7, 8, 9, 13, 14, 21, 23
        Hint for #23 : Find a solution of this
        problem in a Calc. textbook (usually
        in Calc. Ⅲ, under "Velocity & acceleration").
        ~~####~~   # 3
        ~~##~~   # 10 (Hint: proceed similarly to #23)

Sec. 1.3   5, 1, 2, 4, 10.