Lecture 13 Real repeated roots; Reduction of order of a DE.

In Lecture 12 we showed that if a DE-2 has distinct real roots $\lambda_1 \neq \lambda_2$ of the characteristic equation, then

$$\{ y_1 = e^{\lambda_1 t}, \ y_2 = e^{\lambda_2 t} \} \text{ form a FS.}$$

Hence we have "enough building blocks" to construct the solution of any IVP.

Q: What if $\lambda_1 = \lambda_2$?
Then we have only one solution.
Does it form a FS?

A: No.
There is another solution that we yet need to find.

(1) Finding the 2nd solution for the FS of a linear homogeneous DE-2 with constant coefficients.

Let the DE be

$$a y'' + b y' + c y = 0 \qquad (1)$$

$a, b, c = \text{const}$; and $a = 1$ w/o loss of generality.
If we seek $y = e^{\lambda t}$, $\Rightarrow$

$$\lambda^2 + b \cdot \lambda + c = 0, \qquad (2)$$

and we know that this quadratic eqn.

has only one root, say $\lambda = \alpha$.
I.e. (2) is equivalent to

$$(\lambda - \alpha)^2 = 0, \qquad (3a)$$

or

$$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0. \qquad (3b)$$

Thus, we know that the DE must be:

$$y'' - 2\alpha y' + \alpha^2 y = 0. \qquad (4)$$

<u>Trick</u> (similar to variation of parameters; see Lecture 2, topic 3)

$$\boxed{\begin{array}{l}(f\cdot g)' = f'g + f\cdot g' \\ (f\cdot g)'' = (f'g+fg')' = f''g + 2f'g' + fg''\end{array}}$$

Substitute

$$y = e^{\alpha t} \cdot u(t) \qquad (5)$$

into (4):

$$(e^{\alpha t}u)'' - 2\alpha(e^{\alpha t}u)' + \alpha^2(e^{\alpha t}u) = 0$$

$$\underbrace{(e^{\alpha t})''\cdot u + 2(e^{\alpha t})'u' + e^{\alpha t}\cdot u''}_{y''} - 2\alpha\underbrace{\left((e^{\alpha t})'u + e^{\alpha t}\cdot u'\right)}_{y'}$$

$$+ \alpha^2\cdot(e^{\alpha t}u) = 0$$

$$\Rightarrow \underbrace{\left[(e^{\alpha t})'' - 2\alpha\cdot(e^{\alpha t})' + \alpha^2(e^{\alpha t})\right]}_{=0 \text{ because } y=e^{\alpha t} \text{ solves (4)}} +$$

$$+ \left[\cancel{2(e^{\alpha t})'u'} + e^{\alpha t}\cdot u'' - \cancel{2\underbrace{\alpha e^{\alpha t}}_{(e^{\alpha t})'}\cdot u'}\right] = 0$$

$$\Rightarrow \quad e^{\alpha t} \cdot u'' = 0$$

$$\Rightarrow \quad u'' = 0$$

We integrate this twice (Lecture 10, topic 3; or Calc. I or Calc. III ← motion with constant acceleration) and find:

$$u = a_1 t + a_2 , \qquad (6)$$

$a_1, a_2 = \text{const.}$

Combining (5) & (6):

$$y_2 = (a_1 t + a_2)\, e^{\alpha t} \qquad (7)$$

is the 2nd solution of DE (1). Computing Wronskian of $y_1 = e^{\alpha t}$ and y_2 given by (7), one finds

$W \neq 0$ (see p. 128).

Thus, $\{y_1, y_2\}$ form a FS of solutions of (1), and so the general sol'n is:

$$y = c_1 e^{\alpha t} + c_2 (a_1 t + a_2) e^{\alpha t}$$

$$= \underline{c_1 e^{\alpha t}} + c_2 a_1 t e^{\alpha t} + \underline{(c_2 a_2) e^{\alpha t}}$$

$$= \underbrace{(c_1 + c_2 a_2)}_{\mathbb{C}_1} e^{\alpha t} + \underbrace{(c_2 a_1)}_{\mathbb{C}_2} \cdot t e^{\alpha t}$$

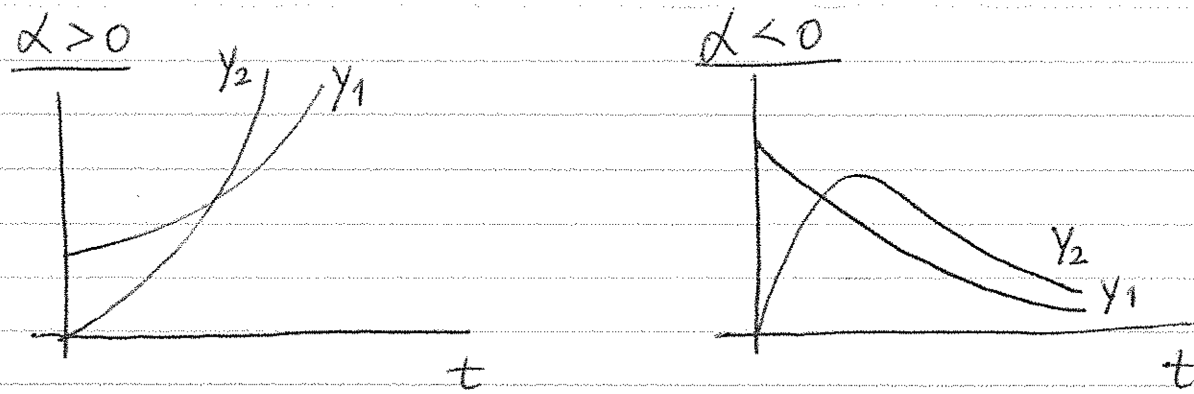
rename: $\mathbb{C}_1$ $\mathbb{C}_2$

$$= \mathbb{C}_1 e^{\alpha t} + \mathbb{C}_2 \cdot t e^{\alpha t}.$$

Thus, we can take as the FS:

$$\boxed{y_1 = e^{\alpha t}, \quad y_2 = te^{\alpha t}} \qquad (8)$$

Note: These solutions look like:



(2) Finding the second solution of a homogeneous linear DE-2; The general case of Reduction of Order.

The same trick as above also works for a general homogeneous linear DE-2:

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad (9)$$

Suppose we know one its sol'n:

$$y = y_1(t). \qquad (10)$$

Then seek $\quad y_2 = y_1(t) \cdot u(t). \qquad (11)$

See Ex. 2 + p. 129 in textbook & the following example.

<u>Ex. 1</u> DE

$$t y'' + (2t-1) y' + (t-1) y = 0 \qquad (12)$$

has a sol'n $y_1 = e^{-t}$, which can be verified by inspection.

(a) Find y_2 from (11) and verify that these y_1, y_2 form a fundamental set for some $t \in (a, b)$. Identify this (a, b).

(b) Solve the IVP consisting of (12) & IC:

$$y(0) = 0, \quad y'(0) = 1 \qquad (13)$$

and explain the result.

<u>Sol'n:</u> (a) 1) $y_2 = e^{-t} \cdot u$, substitute into (12):

$$t\left[e^{-t} u'' + 2 \cdot (e^{-t})' \cdot u' + \underline{(e^{-t})'' \cdot u}\right] +$$

$$(2t-1) \cdot \left[e^{-t} \cdot u' + \underline{(e^{-t})' \cdot u}\right] +$$

$$\underline{(t-1) \cdot e^{-t} \cdot u} = 0 \quad \Rightarrow$$

$$\left\{t e^{-t} u'' + t \cdot 2 (e^{-t})' u' + (2t-1) \cdot e^{-t} \cdot u'\right\} +$$

$$u \cdot \underbrace{\left[t (e^{-t})'' + (2t-1)(e^{-t})' + (t-1) e^{-t}\right]}_{=0 \text{ since } e^{-t} \text{ is a sol'n of } (12)} = 0$$

$$\Rightarrow$$

$$\cancel{e^{-t}} t u'' - 2t\cancel{e^{-t}}u' + (2t-1)\cancel{e^{-t}} \cdot u' = 0$$

$$\Rightarrow$$

$$t u'' - u' = 0 \qquad \Big|\ u' \equiv v$$

$$t v' - v = 0 \quad \Rightarrow$$

$$v = Ct \quad \text{(Lecture 2, Ex. 6)}$$

$$\Rightarrow \quad u = \int Ct\,dt = \frac{c}{2}t^2 + c_0. \quad \text{Thus}$$

$$y_2 = \left(\frac{c}{2}t^2 + c_0\right)e^{-t}.$$

<u>Note:</u> General sol'n:

$$y = \underline{c_1 e^{-t}} + c_2\left(\frac{c}{2}t^2 e^{-t} + \underline{c_0 e^{-t}}\right)$$

$$= \underbrace{(c_1 + c_2 c_0)}_{C_1} e^{-t} + \underbrace{\left(\frac{c_2 c}{2}\right)}_{C_2} \cdot t^2 e^{-t}.$$

Thus, can take $y_1 = e^{-t}$, $y_2 = t^2 e^{-t}$.

2) $$W = \begin{vmatrix} e^{-t} & t^2 e^{-t} \\ -e^{-t} & (2te^{-t} - t^2 e^{-t}) \end{vmatrix}$$

$$= (2te^{-2t} - \cancel{t^2 e^{-2t}}) + (\cancel{t^2 e^{-2t}})$$

$$= 2t \cdot e^{-2t}.$$

$W \neq 0$ for $t > 0$ or $t < 0$.

This makes sense because (12) can be written as

$$y'' + \underbrace{\left(\frac{2t-1}{t}\right)}_{p(t)} y' + \underbrace{\left(\frac{t-1}{t}\right)}_{q(t)} y = 0,$$

and p, q are continuous for $t>0$ or $t<0$.

Thus, $\{e^{-t}, t^2e^{-t}\}$ is a FS for $t \in (0, \infty)$ or $t \in (-\infty, 0)$.

(b) $y(0) = 0 \Rightarrow \left(c_1 e^{-t} + c_2 \cdot t^2 e^{-t}\right)\Big|_{t=0} = 0$

$$c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$y'(0) = 1 \Rightarrow \left(c_1 \cdot (-e^{-t}) + c_2 (2t - t^2) e^{-t}\right)\Big|_{t=0} = 0$

$$-c_1 + c_2 \cdot (0 - 0) = \underline{1}.$$

Thus $\begin{cases} y(0)=0 \\ y'(0)=1 \end{cases} \Rightarrow \begin{cases} c_1 + 0 \cdot c_2 = 0 \\ -c_1 + 0 \cdot c_2 = 1 \end{cases} \Rightarrow$ no solutions!

Why? Because our IC is at $t=0$, but $\{y_1, y_2\}$ is <u>not</u> a FS on $[0, \infty)$ because $W(0) = 0$!

Thus, to be able to solve a IVP, we must take $t_0 \neq 0$.

13-8

HW Sec. 3.4

1, 3, 5, 6, 7, 10 ⇐ DE w/repeated root; IVP

11, 12 — identify DE from graph

13 — rudimentary BVP

15, 17, 14, 20 (think how 20 generalizes 14) ⇐ topic ②

Ans. #12 : $\alpha = 0$, $y_0 = 0$, $y_0' = -1/2$.

(For other even ## use DSolve).