

Lecture 17 Particular sol'n of a non-homogeneous DE-2: the method of undetermined coefficients.

In Lec. 16 we learned that to find the general sol'n of a DE, we need its particular solution. In this Lecture we'll learn one method of finding a part. sol'n. This method works in some "simple" cases. In Lec. 18 we'll learn a more general method that works in all cases.

In all Examples in this Lecture the DE is:

$$ay'' + by' + cy = g(t), \quad (1)$$

$$a, b, c = \underline{\text{const}} \quad (\text{in fact, } a=1).$$

Also, denote:

$$y_h(t) = \text{solution of the } \underline{\text{homogeneous}} \text{ version of (1).}$$

Ex. 1 $\underline{g(t) = e^{\alpha t} \neq y_h(t)}$

$$y'' - y' - 2y = G \cdot e^{\alpha t}, \quad G = \text{const.}$$

Find y_p , given that $e^{\alpha t} \neq y_h(t)$.

Sol'n: 1) Let's first see what $y_h(t)$ is.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -1, 2 \quad (2)$$

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Thus, $\alpha \neq -1$ or 2 .

2) Seek $y_p = Ae^{\alpha t}$

$$y_p' = \alpha Ae^{\alpha t}, \quad y_p'' = \alpha^2 Ae^{\alpha t}$$

Substitute into (1):

$$\alpha^2 Ae^{\alpha t} - \alpha Ae^{\alpha t} - 2Ae^{\alpha t} = Ge^{\alpha t}$$

$$Ae^{\alpha t}(\alpha^2 - \alpha - 2) = Ge^{\alpha t}$$

$$A = \frac{G}{\alpha^2 - \alpha - 2}$$

Note: Since $\alpha \neq -1$ or $2 \Rightarrow$
 $\alpha^2 - \alpha - 2 \neq 0 \Rightarrow$
can find A .

See Ex. 1 in book for ##.

Ex. 2 $g(t) = t^n$ ($\neq y_h(t)$).

Find y_p for

$$y'' - y' - 2y = t^2$$

homogen.

Note 1: t^n is not a sol'n of a DE with const. coefficients for any n .

Sol'n:

Seek $y_p = A_2 t^2 + A_1 t + A_0$ (3)

Q: Why not just At^2 ?

A: a) An explicit reason why this won't work will be clear soon.

b) A general (philosophical) reason is:

$g(t) = t^2$ is a special case of a 2nd-degree polynomial. Then y_p is to be sought in the form of the most general 2nd-degree poly.

$$y_p' = A_2 \cdot 2t + A_1 \cdot 1$$

$$y_p'' = A_2 \cdot 2$$

Substitute into DE:

$$(2A_2) - (2A_2t + A_1) - 2(A_2t^2 + A_1t + A_0) = 1t^2$$

Collect powers of t :

$$\textcircled{t^2} : -2A_2 = 1 \quad \text{(I)}$$

$$\textcircled{t^1} : -2A_2 - 2A_1 = 0 \quad \text{(II)}$$

$$\textcircled{t^0} : 2A_2 - A_1 - 2A_0 = 0 \quad \text{(III)}$$

Note 2: Going back to the Question above, we see that $y_p = A_2 t^2$ (i.e. w/o A_1, A_0) would not have been able to satisfy all of the 3 eqs. (I), (II), (III). Hence we needed 3 coefficients = # of coefficients in the most general 2nd-degree polynomial.

Continuing with the solution:

Find A_2 from (I), then A_1 from (II),
then A_0 from (III):

$$(I) \Rightarrow A_2 = -1/2$$

$$(II) \Rightarrow A_1 = -A_2 = 1/2$$

$$(III) \Rightarrow A_0 = \frac{1}{2}(-1 - 1/2) = -3/4$$

$$\text{Thus } y_p = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4}$$

Note 3: The general sol'n is

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + y_p$$

Ex. 3 $g(t) = \{\sin \beta t \text{ or } \cos \beta t\} \neq y_h(t)$.

$$y'' - y' - 2y = G \sin(\beta t);$$

($G = \text{const}$)

find y_p .

Sol'n: Seek

$$\boxed{y_p = A \sin \beta t + B \cos \beta t} \quad (4)$$

Q: Why not just $A \sin \beta t$?

A: a) An explicit reason will be clear soon.

b) The general (philosophical) reason:

- In Lec. 14 we saw that

$$e^{i\beta t} = \cos \beta t + i \sin \beta t$$

Thus, \cos & \sin are not unrelated functions, but are two "parts" of one exponential function.

- In Lec. 14 we also explored the identity

$$\cos\left[\beta\left(t - \frac{\delta}{\beta}\right)\right] = \cos(\beta t - \delta) = \cos \beta t \cdot \cos \delta + \sin \beta t \cdot \sin \delta,$$

\Rightarrow \cos & \sin are two "parts" of a "shifted \cos ", $\cos \beta\left(t - \frac{\delta}{\beta}\right)$.

Thus, by taking a linear combination (4) of \cos & \sin , we are taking the most general form of the appropriate trig. function.

Continuing with the solution:

$$(4) \Rightarrow y' = \beta A \cos \beta t - \beta B \sin \beta t$$

$$y'' = -\beta^2 A \sin \beta t - \beta^2 B \cos \beta t.$$

Substitute into DE:

$$(-\beta^2 A \sin \beta t - \beta^2 B \cos \beta t) - (\beta A \cos \beta t - \beta B \sin \beta t)$$

$$-2(A \sin \beta t + B \cos \beta t) = G \sin \beta t.$$

Collect terms at $\sin \beta t$ & $\cos \beta t$ separately!

See ##38, 39
in Sec. 3.8
for more details

@ sin!

$$A \cdot [-\beta^2 - 2] + B \cdot [\beta] = G \quad | \cdot \beta$$

$$\textcircled{\text{@ cos:}} \quad A \cdot [-\beta] + B \cdot [-\beta^2 - 2] = 0 \quad | \cdot (-\beta^2 - 2)^+$$

Multiply as shown above and add:

$$A \cdot \cancel{0} \xrightarrow{\text{by design}} + B \cdot [\beta^2 + (-\beta^2 - 2)^2] = G \cdot \beta$$

$$\Rightarrow B = G\beta / [\beta^2 + (-\beta^2 - 2)^2]$$

similarly:

$$A = G(-\beta^2 - 2) / [\beta^2 + (-\beta^2 - 2)^2].$$

Note 1: It is clear that with just $y_p = A \sin \beta t$ we would not have been able to satisfy two eqs. to match the sin & cos terms.

Note 2: Since $\beta^2 + (-\beta^2 - 2)^2 \neq 0$ (in fact, it is > 0), we can solve for A & B no matter what G and β are. See Ex. 3 in book for numbers.

Ex. 4 $g(t) = e^{\alpha t} = y_h(t).$

Find y_p for

$$y'' - y - 2y = G e^{2t}.$$

Note 1: e^{2t} is one of the hom. sol'ns, since $\lambda_{1,2} = -1, 2$.

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So, if we try to substitute $y_p = Ae^{2t}$, as in Ex. 1, we'll get:

$$A \underbrace{(2^2 - 2 - 2)}_0 e^{2t} = Ge^{2t}$$

$A \cdot 0 = G \Rightarrow$ no solutions.

Sol'n: Trick: seek

$$\boxed{y_p = A \cdot t \cdot e^{2t}} \quad (5)$$

$$y_p' = 2A \cdot t \cdot e^{2t} + A \cdot e^{2t}$$

$$y_p'' = (2 \cdot A \cdot t \cdot e^{2t} + 2A \cdot e^{2t}) + 2Ae^{2t}$$

Substitute into DE:

$$A \left[(2 \cdot t \cdot e^{2t} + 2 \cdot 2e^{2t}) - (2te^{2t} + e^{2t}) - 2te^{2t} \right] = Ge^{2t}$$

$$A \cdot \left[\underbrace{(2^2 - 2 - 2)}_{=0} \cdot te^{2t} + (4 - 1)e^{2t} \right] = Ge^{2t}$$

$$A \cdot 3 \cdot e^{2t} = Ge^{2t} \Rightarrow A = G/3$$

$$\text{Thus, } y_p = \frac{G}{3} te^{2t}$$

Note 2: The above trick is similar to that used in Lec. 13 and earlier in Lec. 2:

$$\boxed{y_{\text{new}} = y_{\text{old}} \cdot u(t)} \quad (6)$$

In fact, substituting (6) into DE, one can verify that in this case, $u(t) \equiv t$ indeed.

Note 3: When the DE has repeated λ (as in Lec. 13), a more general substitution than (5) will be needed. See Ex. 5 in book. Note that substitution (6) will still give an appropriate form of $u(t)$, but with more work.

Note 4: On your own:



- MUST READ EX. 6 (a, c, d) / book;
- Use Table on p. 163 to find the appropriate form of y_p (see also Ex. 6 in these Notes below).

Ex. 5 $g(t) = \{ \sin pt \text{ or } \cos pt \} = y_p(t)$

Find y_p for

$$y'' + \omega^2 y = G \sin \omega t.$$

(Here, $g(t) = \sin \omega t$ is a sol'n of the homogeneous DE; see Lec. 10 & 14.)

Sol'n: Combining the ideas of Ex. 3 and Ex. 4, we seek:

$$y_p = t \cdot (A \sin \omega t + B \cos \omega t) \quad (7)$$

$$y_p' = t\omega(A\cos\omega t - B\sin\omega t) + (A\sin\omega t + B\cos\omega t)$$

$$y_p'' = -t\omega^2(A\sin\omega t + B\cos\omega t) + 2\omega(A\cos\omega t - B\sin\omega t)$$

Substitute into DE:

$$\underline{-t\omega^2(A\sin\omega t + B\cos\omega t) + 2\omega(A\cos\omega t - B\sin\omega t)} + \omega^2 \underline{[t(A\sin\omega t + B\cos\omega t)]} = G\sin\omega t$$

The underlined terms exactly cancel, \Rightarrow

$$2\omega(A\cos\omega t - B\sin\omega t) = G\sin\omega t$$

@ sin ωt : $-2\omega B = G \Rightarrow B = -\frac{G}{2\omega}$

@ cos ωt : $2\omega A = 0 \Rightarrow A = 0$

Thus

$$y_p = -\frac{G}{2\omega} \cdot t \cdot \cos\omega t$$

Important Note: This Example describes an important physical phenomenon known as the resonance. It describes a solution whose amplitude grows in time,

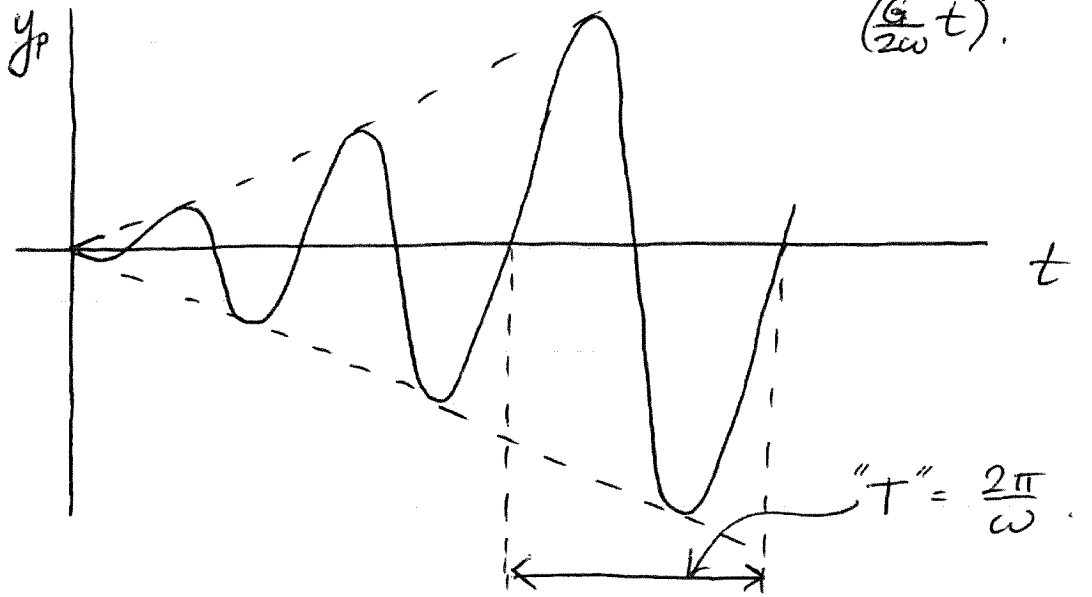
even though the r.h.s. $g(t) = G\sin\omega t$ does not grow.

This happens only when:

- (y') -term is absent; and
- $(\omega$ on rhs) equals $(\omega$ on l.h.s.)

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Plot:



HW: General note: see Table on p. 163 for the form of y_p in each case.

1, 3, 4 \leftarrow find y_p ; $g \neq y_h$

5, 11, 13 \leftarrow find y_p , $g = y_h$ (exponential)

17, 19 \leftarrow general form of y_p

29, 30 \leftarrow given y_p & g , find α, β in $y'' + \alpha y' + \beta y = g$.

31 \leftarrow given asymptotic behavior for $g \neq y_h$, find IC.

Find y_p :

WP1 $y'' + \omega^2 y = \cos \omega t$

WP2 $y'' + 2\alpha y' + \omega^2 y = \cos \omega t$

WP3 $y'' + \omega^2 y = \cos(0.9\omega t)$

Plot all three sol'n together (use Mathematica) for:

$\omega = 1, \alpha = 0.1; t \in [0, 10\pi]$.