Lecture 18   Particular sol'n of a
nonhomogeneous DE-2:
the method of variation of parameters.

(1)   Brief review of DE-1 case

Solve $y' + p(t)y = g(t)$ (1)
given y_h: $y_h' + p(t)y_h = 0$.   (2)

Solution: seek $y = y_h \cdot u(t)$ $\Rightarrow$

$(y_h \cdot u' + \underbrace{y_h' u) + p \cdot y_h \cdot u}_{= 0 \text{ by } (2)} = g$

$\Rightarrow \quad y_h \cdot u' = g \quad \Rightarrow \quad u' = g/y_h$

$\Rightarrow$ solve for u $\Rightarrow$ $y = y_h \cdot u$. $\checkmark$

(2)   Formula for DE-2 case

Solve $y'' + p(t)y' + q(t)y = g(t)$ (3)

given that $\{y_1, y_2\}$ form a FS of the

homogeneous DE-2

$$y'' + py' + qy = 0.   (4)$$

Use a similar trick: seek

$$\boxed{y_p(t) = y_1 \cdot u_1(t) + y_2 \cdot u_2(t)},   (5)$$

where $u_1(t)$, $u_2(t)$ are TBD.

$$y_p' = (y_1' u_1 + y_1 u_1') + (y_2' u_2 + y_2 u_2')$$

(new)

rearrange:

$$\overset{\vee}{=} [y_1' u_1 + y_2' u_2] + [y_1 u_1' + y_2 u_2']$$

Before we compute y_p'', use Trick #2 :

Require :

$$\boxed{y_1 u_1' + y_2 u_2' = 0} \qquad (6)$$

Continue using (6) :

$$y_p' = [y_1' u_1 + y_2' u_2] \qquad \Rightarrow$$

$$y_p'' = y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2' \, .$$

Substitute into (3) :

$$(y_1'' u_1 + y_1' u_1' + \underline{y_2'' u_2} + y_2' u_2') + p \cdot (\underline{y_1' u_1 + y_2' u_2})$$

$$+ \; q \cdot (\underline{y_1 u_1} + \underline{y_2 u_2}) = g(t)$$

Rearrange:

$$\overset{0 \longleftarrow \text{by (4)} \longrightarrow 0}{(y_1'' + p y_1' + q y_1) \cdot u_1 + (y_2'' + p y_2' + q y_2) \cdot u_2}$$

$$+ [y_1' u_1' + y_2' u_2'] = g$$

$$\Rightarrow \qquad \boxed{y_1' u_1' + y_2' u_2' = g} \qquad (7)$$

__Note 1:__ Had we not required (6), then y_p'' would have had terms like $y_1 u_1''$ & $y_2 u_2''$, and (7) would not have had its simple form.

However, a __systematic__ explanation for trick #2 will be given only in Chap. 4.

Continuing with the solution:

(6): $\quad y_1 u_1' + y_2 \cdot u_2' = 0 \qquad \bigg| \cdot y_2'$

(7): $\quad y_1' u_1' + y_2' u_2' = g \cdot \qquad \bigg| \cdot y_2$

This is a __linear system__ for $\{u_1', u_2'\}$:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \qquad (8)$$

Its solution exists for __any__ g because

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv W(t) \neq 0, \quad \text{because } \{y_1, y_2\} \text{ is a FS of sol'ns.}$$

Solve (6) & (7).

E.g., $(6) \cdot y_2' - (7) \cdot y_2$ (see above) yields:

$$(y_1 y_2' - y_1' y_2) u_1' + (\overbrace{y_2 y_2' - y_2' y_2}^{0}) u_2 = 0 - g \cdot y_2$$

$$\Rightarrow \boxed{u_1' = \dfrac{-g \cdot y_2}{W(t)}} \qquad (9a)$$

Similarly:

$$u_2' = \frac{g \cdot y_1}{W(t)}$$ (9b)

Integrating:

$$u_1(t) = - \int^t \frac{g(s) \, y_2(s)}{W(s)} \, ds$$ (10a)

$$u_2(t) = \int^t \frac{g(s) \, y_1(s)}{W(s)} \, ds$$ (10b)

Combine (10) & (5):

$$y_p(t) = -y_1(t) \int^t \frac{g(s) \, y_2(s)}{W(s)} \, ds + y_2(t) \int^t \frac{g(s) \, y_1(s)}{W(s)} \, ds$$ (11)

Note 2: The upper limit "t" above indicates that after the integration, we should set the variable $s = t$.
The lower limit is not important because it will only shift the integral by a constant. Then sol'n (11) will shift by:

$$a_1 y_1 + a_2 y_2 \equiv y_c(t) \leftarrow \text{sol'n of homogeneous DE,}$$

and changing y_p by some $y_c(t)$ does not affect the general solution

$$y(t) = y_p(t) + C_1 y_1 + C_2 y_2.$$ (12)

③ Examples

Ex. 1 $y'' - y' - 2y = Ge^{2t}$

Find y_p.

Note 1: We solved this in Lec. 17 by a substitution

$$y_p = A \cdot t e^{2t} \qquad (13)$$

(because $y_h = e^{2t}$ is a sol'n of the homogeneous problem).

However, this substitution was postulated but not derived.

Now we will derive (13).

Sol'n:

1) Ex. 4 in Lec. 17 $\Rightarrow$

$$y_1 = e^{-t}, \quad y_2 = e^{2t}; \quad W(t) = 3e^{t} \quad (p.124 \text{ of book})$$

We need $u_1(t)$, $u_2(t)$.

Let's compute $u_2(t)$ and show that term

$$y_2 \cdot u_2$$

agrees with (13)

Note 2: In HW 18 you may explore what happens to term $y_1 u_1$.

2) From formula (10b):

$$u_2(t) = \int^{t} \frac{g(s)\, y_1(s)}{W(s)}\, ds = \int^{t} \frac{Ge^{2s} \cdot e^{-s}}{3e^{s}}\, ds =$$

$$= \int^t \frac{G}{3}\, ds = \frac{G}{3} s \Big|^t_{t_0} = \frac{G}{3} t - \frac{G}{3} t_0.$$

↳ some const

Thus,

$$y_2 \cdot u_2 = y_2 \cdot \underbrace{\left(\frac{G}{3} t - \frac{G}{3} t_0\right)}_{u_2} =$$

$$= \underbrace{\frac{G}{3} t e^{2t}}_{\text{same as in Ex. 4 of Lec. 17}} + \underbrace{\left(-\frac{G}{3} t_0\right) \cdot e^{2t}}_{\substack{\text{some} \\ \text{const}}}$$

part of the
homogeneous
sol'n.

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Ex. 2   Find the general sol'n of

$$t^2 y'' - t y' + y = t$$

Sol'n:

1.) Always put into form

(coefficient) → $y'' + p y' + q y = g$,
= 1

because previous formulas were
derived for this form (i.e. when
the coefficient of y'' is <u>1</u>).

So:

$$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = \frac{1}{t}$$

$$\Rightarrow g(t) = 1/t.$$

2) In #15 of HW13 you showed that

$$y_1 = t, \quad y_2 = t \ln t$$

form a fundamental set of sol'ns for the homogeneous version of this DE.

$$W(t) = \begin{vmatrix} t & t\ln t \\ 1 & (\ln t)+1 \end{vmatrix} = t\ln t + t - t\ln t = t$$

($\neq 0$ for $t \neq 0$).

3) (10a) $\Rightarrow$ $u_1(t) = -\int^t \dfrac{g(s)\, y_2(s)}{W(s)}\, ds$

$$= -\int^t \frac{\frac{1}{s} \cdot s \ln s}{s}\, ds = -\int^t \frac{\ln s}{s}\, ds \qquad \begin{aligned} v &= \ln s \\ dv &= ds/s \end{aligned}$$

$$= -\int^{s=t} v \cdot dv = -\frac{v^2}{2}\Big|^{s=t} = -\frac{(\ln s)^2}{2}\Big|^{s=t} = -\frac{(\ln t)^2}{2}.$$

(10b) $\Rightarrow$

$$u_2 = \int^t \frac{g(s)\, y_1(s)\, ds}{W(s)} = \int^t \frac{\frac{1}{s} \cdot s \cdot ds}{s} = \int^t \frac{ds}{s}$$

$$= \ln|s| \,\Big|^{s=t} = \ln|t| \xrightarrow[\text{assume } t>0]{} \ln t.$$

4)

$$y_p = -\frac{(\ln t)^2}{2} \cdot t + \ln t \cdot t\ln t = \frac{1}{2} t (\ln t)^2$$

General sol'n:

$$y(t) = c_1 t + c_2 t\ln t + \frac{1}{2} t (\ln t)^2.$$

Ex. 3 . Find y_p and then solve the IVP :

$$y'' + \omega^2 y = g(t).$$

$$y(0) = y_o, \quad y'(0) = y_o'.$$

Sol'n: i)   Lec. 10 (also Lec. 14) $\Rightarrow$

$y_1 = \cos \omega t$, $y_2 = \sin \omega t$ form a FS;
$W(t) = \omega \ (= \text{const})$.

2) $u_1 = -\int_a^t \dfrac{g(s) y_2(s)}{W(s)} ds$

$\qquad\qquad\qquad\qquad$ — some constant.

Let us choose (a) so that $y_p(0) = 0$;
then c_1, c_2 in $y_c(t)$ will be chosen
to satisfy the IC.

Then $a = 0$, because $\int_0^{t=0} (\text{anything}) = 0$.

$u_1(t) = -\int_0^t \frac{1}{\omega} \cdot g(s) \cdot \sin(\omega s) ds \equiv -\frac{1}{\omega} \int_0^t g(s) \sin(\omega s) ds$

$u_2(t) = \int_0^t \frac{1}{\omega} g(s) \cos(\omega s) ds$.

(✱) $y_p(t) = \frac{1}{\omega} \left[-\cos(\omega t) \cdot \int_0^t g(s) \cdot \sin(\omega s) ds + \sin(\omega t) \cdot \int_0^t g(s) \cos(\omega s) ds \right]$

$\qquad = \frac{1}{\omega} \int_0^t g(s) \left[\sin(\omega t) \cos(\omega s) - \sin(\omega s) \cdot \cos(\omega t) \right] ds$

trig
identity $\stackrel{\text{}}{=} \frac{1}{\omega} \int_0^t g(s) \sin\big(\omega[t-s]\big) ds$.

General: $y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s]) ds$.

3) $y(0) = y_0 \Rightarrow$

$c_1 \cdot 1 + c_2 \cdot 0 + \underset{\nearrow 0 \text{ (see above)}}{y_p(0)} = y_0 \Rightarrow c_1 = y_0.$

$y'(0) = y_0' \Rightarrow$

$c_1 \cdot (-\omega) \cdot 0 + c_2 \cdot \omega \cdot 1 + y_p'(0) = y_0'.$

To find $y_p'(0)$, one could have differentiated Eq. (*) on previous page. For that, one needs:
- product rule, and
- Fund. Thm. of Calculus.

However, this is tedious.

Fortunately, there is another "Rule" of differentiation, which works here, as well as in many other circumstances. It is based on a composition of two familiar Rules.

● Familiar Rule 1:
$$\frac{d}{dt} \int_a^b f(t,s)\, ds = \int_a^b \frac{\partial f(t,s)}{\partial t}\, ds$$

● Familiar Rule 2 (Fund. Thm. of Calculus):
$$\frac{d}{dt} \int_a^t h(s)\, ds = h(t).$$

$\longrightarrow$ <u>NEW RULE</u>:

$$\frac{d}{dt} \int_a^t f(t,s)\, ds = \underbrace{\int_a^t \frac{\partial f(t,s)}{\partial t}\, ds}_{\text{from F. Rule 1}} + \underbrace{f(t,t)}_{\text{from F. Rule 2}} \tag{14}$$

Apply this NEW RULE to $y_p'(t)$:

$$y_p'(t) = \frac{1}{\omega} \int_0^t g(s) \cdot \omega \cdot \cos(\omega[t-s])ds + \frac{1}{\omega} \cdot g(t) \cdot \sin(\omega[t-t])$$

$$= \int_0^t g(s) \cos(\omega[t-s])ds.$$

Thus $y_p'(0) = \int_0^0 \ldots = 0.$

Note 1: With the $y_p(t) = \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s])ds$, one has convenient conditions:

$$y_p(0) = 0 \quad \text{and} \quad y_p'(0) = 0.$$

Thus, coming back to the top of p. 18-9:

$$y'(0) = y_0' \implies 0 + c_2 \cdot \omega + 0 = y_0' \implies c_2 = \frac{y_0'}{\omega}.$$

Thus the solution of the IVP is:

$$y(t) = y_0 \cos \omega t + \frac{y_0'}{\omega} \sin \omega t + \frac{1}{\omega} \int_0^t g(s) \sin(\omega[t-s])ds.$$

Note 2: An integral of the form

$$\int_0^t f(t-s) g(s) ds$$

is called a convolution of f and g.
We will (hopefully) encounter it again
in Chap. 5.

HW.   Sec. 3.9

$1 \leftarrow$ ($g = $const); compare to method of Lec. 15. 16.

$3, 4, 5, 7, 10, 13 \leftarrow$ find $Y_c, Y_p, Y_{general}$.

$16, 18 \leftarrow$ from given form $\quad y = y_c + \int K(t-s) g(s) ds \quad$ find α, β, Y_0, Y_0' in $\quad y'' + \alpha y' + \beta y = g$.

WP: Solve   IVP $\quad y'' + \omega^2 y = G \cdot \sin \omega t \quad$ using method of Ex. 3 from Notes.   Hint: use $\quad \sin a \cdot \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

Note for #4:   Can do both int'ls w/ Mathematica. (???)

If you want to do them by hand, use subst. $e^t = u$; then ~~some~~ one of them will require a part. fr. expansion → look it up!

Note for #5:   2) Compare the so-obtained particular sol'n with that obtained by method from Lec. 17. How much do they differ by?

1) This will show you what happens to term $Y_1 U_1$ (see Ex. 1 in class)

Note for #7:   Again, compare your Y_p with that from method of und. coeff.

Note for #13:   1) Denote $(t-1) = r$; relate dy/dt and dy/dr, and solve the entire problem in terms of $y(r)$.

2) Will need to find $Y_2(r)$ by method of Lec. 13.

Note for #18:   Use the form $\quad y = c_1 y_1 + c_2 y_2 + y_1 u_1 + y_2 u_2$ and the expressions for u_1, u_2.

Answer: $\quad \alpha = \beta = 0, \quad Y_0 = 0, \quad Y_0' = 1$.