

Lecture 19. Resonance in undamped and damped linear oscillator models

① Exact resonance: review of known results

In Lec. 17 (Ex. 5) we showed that when the external force (proportional to function $g(t)$) has the same frequency

ω as the oscillator's own frequency:

$$y'' + \omega^2 y = G \sin \omega t \quad (1)$$

oscillator's own frequency

frequency of external force

then the particular solution grows with time:

$$y_p = -\frac{G}{2\omega} t \cos \omega t \quad (2)$$

growing "envelope".

You reobtained this (~~or~~ very similar) result in the HW #18.

This phenomenon, namely:

(if frequency of external force = own frequency of oscillator) \Rightarrow (undamped oscillations grow with time)

is called the resonance.

This is a very important phenomenon in diverse areas of science and engineering.

When (frequency of external force) = (own frequency of oscillator), we say that the force is at resonance.

Below we'll explore situations when the force is near resonance but not at resonance, or when a damping is present in the oscillator.

② Non-resonant force: phenomenon of "beats"

We will first consider an oscillator w/o damping:

$$y'' + \omega_0^2 y = F \cos \omega_1 t \quad (3)$$

↑
own frequency
of oscillator

↑
external force
frequency

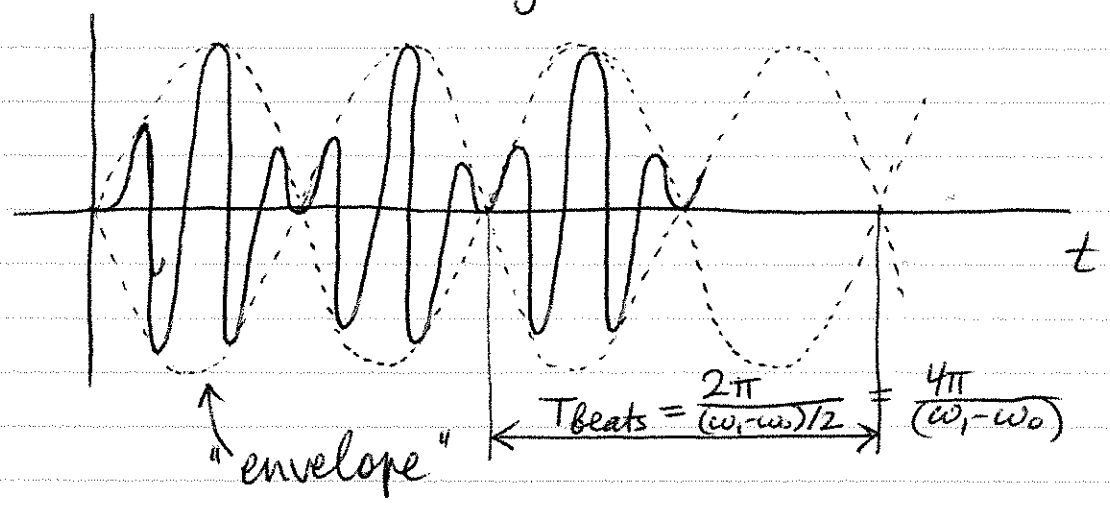
For simplicity, let us assume zero IC:

$$y(0) = 0, \quad y'(0) = 0 \quad (4)$$

By the method of Lec. 17, we have (see Word Problem 3 in HW 17):

$$y(t) = \frac{-F}{\omega_0^2 - \omega_1^2} (\cos \omega_0 t - \cos \omega_1 t) \quad (5)$$

It is shown in Fig. 3.14 in Sec. 3.10:



It is a sinusoidal function modulated by an envelope. To understand this shape, use a trig. identity:

$$\cos(\omega_0 t) - \cos(\omega_1 t) = 2 \sin\left(\frac{\omega_1 - \omega_0}{2} t\right) \cdot \sin\left(\frac{\omega_1 + \omega_0}{2} t\right)$$

($\cos(a-b) - \cos(a+b) = 2 \sin a \cdot \sin b$)

The term $2 \sin\left(\frac{\omega_1 - \omega_0}{2} t\right)$ is circled and labeled "envelope". The term $\sin\left(\frac{\omega_1 + \omega_0}{2} t\right)$ is underlined and labeled "oscillations inside the envelope".

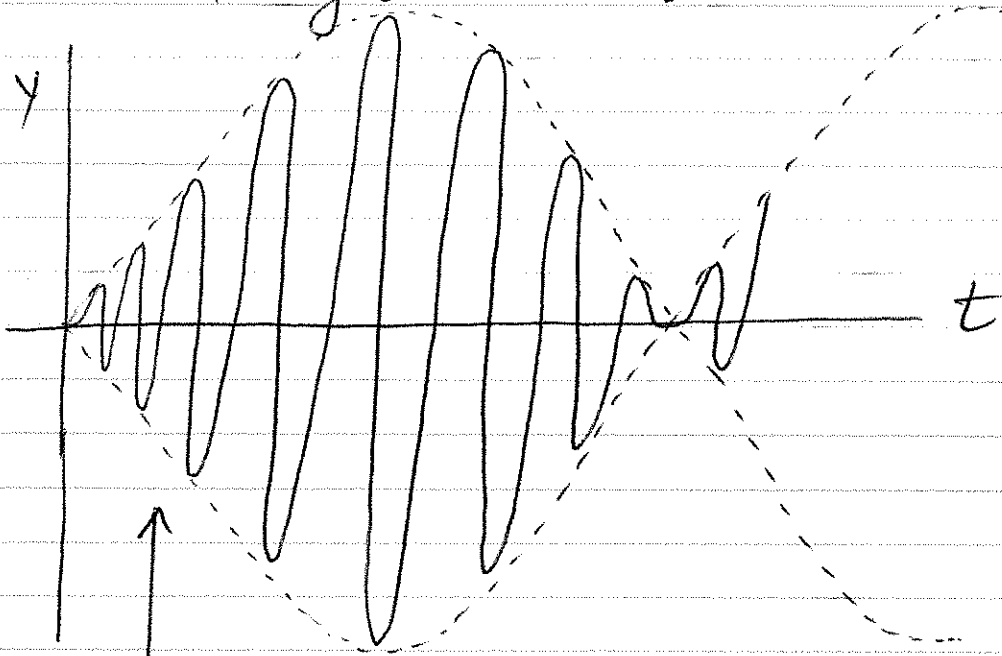
Note 1 As $\omega_1 \rightarrow \omega_0$, two things occur:

1) Amplitude of the oscillations grows as:

$$R \sim \frac{F}{|\omega_1^2 - \omega_0^2|} \rightarrow \infty \text{ as } \omega_1 \rightarrow \omega_0.$$

2) Period of the beats increases as $\frac{4\pi}{|\omega_1 - \omega_0|}$.

For ω_1 very close to ω_0 :



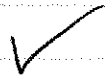
Thus, initially, we have growing oscillations, just as in the exact resonance.

Note 2 We can see by the L'Hospital's Rule that as $\omega_1 \rightarrow \omega_0$, the solution (5) $\rightarrow \frac{F}{2\omega_0} t \cdot \sin \omega_0 t$ (WP3 in HW17):

$$\begin{aligned}
 \lim_{\omega_1 \rightarrow \omega_0} (5) &= \lim_{\omega_1 \rightarrow \omega_0} \frac{-F}{\omega_1^2 - \omega_0^2} (\cos \omega_1 t - \cos \omega_0 t) \\
 &= \lim_{\omega_1 \rightarrow \omega_0} \left(\frac{-F}{\omega_1 + \omega_0} \right) \cdot \frac{\cos(\omega_1 t) - \cos(\omega_0 t)}{\omega_1 - \omega_0} = \\
 &= \lim_{\omega_1 \rightarrow \omega_0} \frac{-F}{\omega_1 + \omega_0} \cdot \lim_{\omega_1 \rightarrow \omega_0} \frac{\cos(\omega_1 t) - \cos(\omega_0 t)}{\omega_1 - \omega_0} = \\
 &= \frac{-F}{2\omega_0} \cdot \left. \frac{d}{d\omega_1} (\cos \omega_1 t) \right|_{\omega_1 = \omega_0}
 \end{aligned}$$

19-5

$$= \frac{F}{2\omega_0} \cdot (t \cdot \sin \omega_0 t) = \frac{F}{2\omega_0} \cdot t \cdot \sin \omega_0 t.$$



Note 3 Of course, physically, a solution describing oscillations whose amplitude $\rightarrow \infty$ as $t \rightarrow \infty$ makes sense only up to some finite t_{\max} . Beyond it, the assumptions of the model get (gradually) violated, and the amplitude saturates at some large but finite value. We'll examine this now, when small damping arrests the growth of amplitude.

③ Effect of damping in the presence of resonant force.

In the notations of Lec. 15: ^{same!}

$$y'' + 2\alpha y' + \omega_0^2 y = F \cos \omega_0 t \quad (6)$$

Damped linear oscillator with own frequency ω_0 .
(In book, " α " is denoted as " δ " (delta).)

"resonant" force
(same frequency as the oscillator's)

Assume zero initial conditions:

$$y(0) = 0, \quad y'(0) = 0. \quad (7)$$

By Lec. 16,

$$y = y_p + y_c$$

$$= \underbrace{(A_1 \sin \omega_0 t + B_1 \cos \omega_0 t)}_{\text{from } y_p} + e^{-\alpha t} \underbrace{(A_2 \sin \beta t + B_2 \cos \beta t)}_{\text{from } y_c}$$

By Lec. 17, since $\sin \omega_0 t$ & $\cos \omega_0 t$ are not homogeneous sol'ns

By Lec. 15, where $\beta = \sqrt{\omega_0^2 - \alpha^2}$

In a HW problem you will verify that $B_1 = B_2 = 0$ for the IC. (7), and that

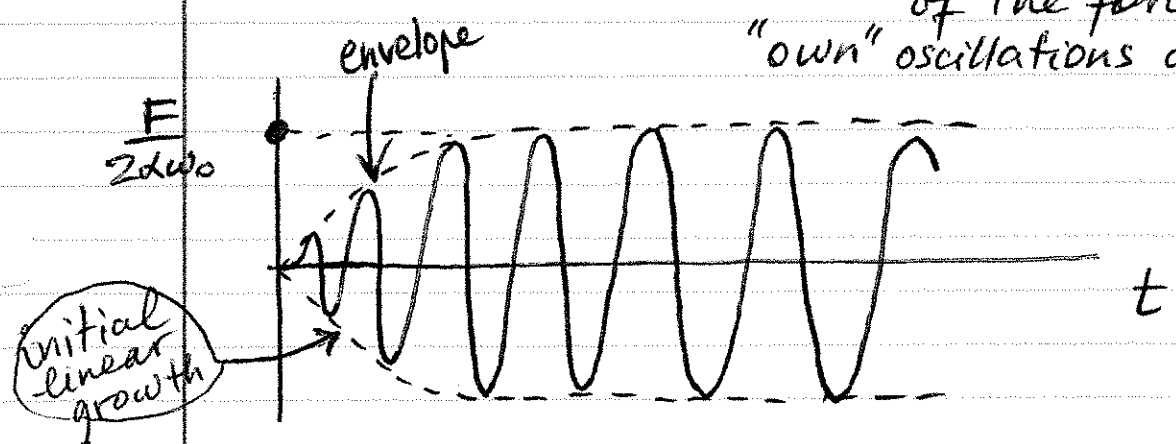
$$y = \frac{F}{2\alpha} \left(\underbrace{\frac{\sin \omega_0 t}{\omega_0}}_{\text{from } y_p} - e^{-\alpha t} \underbrace{\frac{\sin \beta t}{\beta}}_{\text{from } y_c} \right) \quad (8)$$

WP 2 of HW 17

Key property of this solution:

As $t \rightarrow +\infty$, $y_c \rightarrow 0$ as $e^{-\alpha t} \rightarrow 0$, \Rightarrow

$$y(t \rightarrow +\infty) \rightarrow \underbrace{\left(\frac{F}{2\alpha\omega_0} \right)}_{\text{amplitude}} \underbrace{\sin \omega_0 t}_{\text{only oscillations with the frequency of the force remain; "own" oscillations die out.}} \quad (9)$$



④ Damping and non-resonant force

$$y'' + 2\alpha y' + \omega_0^2 y = F \cos \omega_1 t \quad (10)$$

$\omega_1 \neq \omega_0$

The form of the solution is still

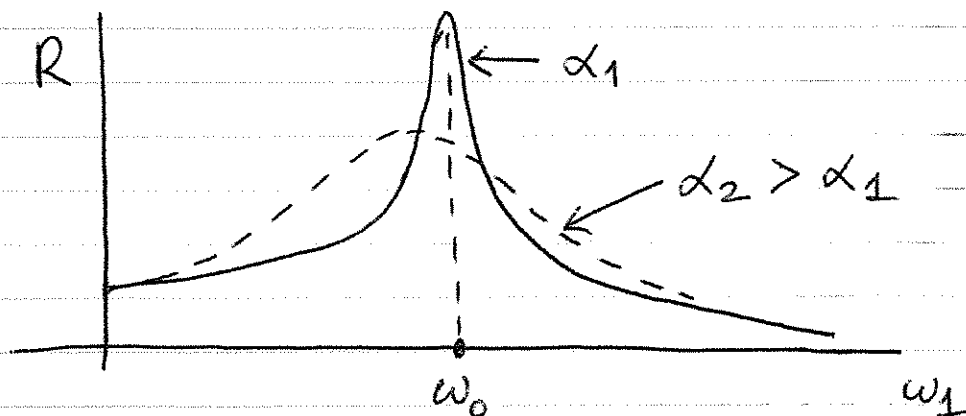
$$y = y_p + y_c,$$

but the coefficients are much more cumbersome: see Eq. (126) on p. 180. The plot is also not as simple as for $\omega_1 = \omega_0$ (see p. 181).

Essential feature:

As $t \rightarrow +\infty$, $y(t) \rightarrow y_p = R \cdot \cos(\omega_1 t - \delta)$,
 since $y_c \sim e^{-\alpha t} \rightarrow 0$

$$R = \frac{F}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + (2\alpha\omega_1)^2}} \quad (11)$$



Essential features of non-resonant solution in presence of damping:

1) For $\alpha_1 < \alpha_2$, solution has larger amplitude for $\omega_1 \approx \omega_0$,

2) but a smaller amplitude for ω_1 far from ω_0 .

("The resonance is narrower when damping is smaller.")

3) The frequency of the peak is $\approx \sqrt{\omega_0^2 - 2\alpha^2}$, i.e. is left-shifted.

The greater the damping, the greater the left-shift of the peak.

HW: Sec. 3.10 ##

2, 3, 4, 6 \leftarrow no damping
resonance \nearrow

10(a,c), 12(b,c) [Accept Eq. (12b) on p. 180 as given],
9 \leftarrow res. force in presence of damping.

#2: $\frac{t}{10} \sin 10t$