Lecture 22 Systems of 1st-order linear DEs : Introduction

① Motivation

We've mostly studied 1st-order & 2nd-order DEs for one variable y. They describe the rate of change or acceleration of one quantity:

$$\frac{dy}{dt} = \ldots \quad \leftarrow \text{DE-1, rate of change}$$

$$\frac{d^2y}{dt^2} = \ldots \quad \leftarrow \text{DE-2, acceleration}$$

However, in real world, things rarely change in isolation from one another. E.g., quantity y may depend on z and vice versa. Then we will have:

$$\frac{dy}{dt} = f_1(y, z)$$

$$\frac{dz}{dt} = f_2(y, z)$$

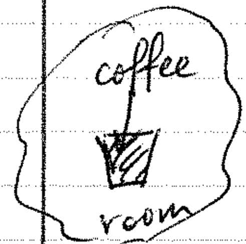
Ex. 1 Set up a system of 1st-order DEs describing heating/cooling of a house with two "rooms".

Preliminaries

In Lec. 1 (Ex. 2(b)) and HW 4 we

considered Newton's Cooling Law:

$$\frac{dT}{dt} = -K(T - T_{room}) = K(T_{room} - T)$$



This law is actually a corollary of the <u>balance of heat</u> (which is similar to balance equation for mixing problems in Lec. 4):

"amount of heat" (i.e., energy)

$$\frac{dQ}{dt} = \underbrace{\begin{pmatrix}\text{Heat}\\\text{flow}\\\text{in}\end{pmatrix} - \begin{pmatrix}\text{Heat}\\\text{flow}\\\text{out}\end{pmatrix}}_{m\cdot(T_{external} - T)} \qquad (1)$$

some constant

$\underline{T_{ext} > T} \Rightarrow$ flow is <u>in</u> (>0)

$\underline{T_{ext} < T} \Rightarrow$ flow is <u>out</u> (<0)

To supplement this equation, we need to relate "amount of heat" with temperature. From thermodynamics:

$$\underbrace{\Delta T}_{} = c \cdot \underbrace{\Delta Q}_{}$$

change in temperature caused by ΔQ

heat capacity (units = $1°F/BTU$)

change in amount of heat (by adding or subtracting heat)

usually measured in BTU (British thermal unit = amount of heat needed to increase temp. of 1 lb of H_2O by $1°F$).

Then (1) becomes

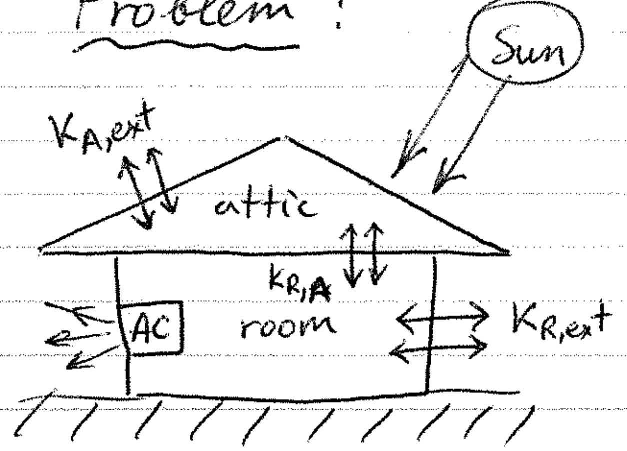
$$\frac{1}{c}\cdot\frac{dT}{dt} = m\,(T_{ext} - T)$$

$$\Rightarrow \quad \frac{dT}{dt} = \underbrace{(cm)}_{\text{"K" of Newton's Cooling Law}} \cdot (T_{ext} - T) \qquad (2a)$$

or

$$\boxed{\frac{dQ}{dt} = \left(\frac{K}{c}\right)\cdot(T_{ext} - T)} \qquad (2b)$$

<u>Problem</u> :



- Sun heats up the attic in a house at the rate of F_{sun} BTU/hr.
- The outside temperature is $T_{ext} = 100°F$.
- The constants K in Newton's Law are:

  $K_{A,ext}$ — between Attic & Outside
  $K_{R,ext}$ — between Room & Outside
  $K_{R,A}$ — between Room & Attic

- The air conditioner in the room removes

F_{AC} BTU/hr of heat.

- The heat capacity of the air (in room & attic) is c.

Set up a system of equations for the evolution of the temperatures T_A & T_R in the Attic and Room.

Sol'n:

Balance eq. for Attic:

$$\frac{dQ_A}{dt} = \underset{\substack{\text{heat flow} \\ \text{from Sun}}}{F_{sun}} + \underbrace{\frac{K_{A,ext}}{c}\left(T_{ext} - T_A\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Attic \& Outside}}} + \underbrace{\frac{K_{R,A}}{c}\left(T_R - T_A\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Attic \& Room}}}$$

Using $dQ_A/dt = d(T_A/c)/dt$ we get:

$$\boxed{\frac{dT_A}{dt} = C \cdot F_{sun} + K_{A,ext}\left(T_{ext} - T_A\right) + K_{R,A}\left(T_R - T_A\right)} \quad (3a)$$

Balance for Room:

$$\frac{dQ_R}{dt} = -\underset{\substack{\text{heat removal} \\ \text{by AC}}}{F_{AC}} + \underbrace{\frac{K_{R,ext}}{c}\left(T_{ext} - T_R\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Room \& Outside}}} + \underbrace{\frac{K_{R,A}}{c}\left(T_A - T_R\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Room \& Attic}}}$$

Using $dQ_R/dt = d(T_R/c)/dt$ we get:

$$\boxed{\frac{dT_R}{dt} = -c \cdot F_{Ac} + K_{R,ext}(T_{ext} - T_R) + K_{R,A}(T_A - T_R)} \quad (3b)$$

Eqs. (3a) & (3b) form the desired system of eqs.

Example 1 in book (Sec. 4.1) considers a mixing problem in two connected tanks, leading to a similar system.

In the most general case, a system of two linear, 1st-order DEs has the form:

$$\begin{aligned} y_1' &= p_{11}(t)\, y_1 + p_{12}(t)\, y_2 + g_1(t) \\ y_2' &= p_{21}(t)\, y_1 + p_{22}(t)\, y_2 + g_2(t) \end{aligned} \quad (4)$$

or in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad (5a)$$

or, in vector-matrix notations:

$$\vec{y}\,' = \underset{\substack{\uparrow \\ \text{matrix}}}{P(t)}\, \vec{y} + \vec{g}(t). \quad (5b)$$

For n lin. DEs, the system has the same form (5b).

2) Auxiliary facts about algebra and Calculus of vectors & matrices.

2a Algebra

0. (A is invertible (i.e. A^{-1} exists)) $\Leftrightarrow$ ($\det A \neq 0$).
1. $\det(AB) = \det(A) \cdot \det(B)$
2. $\det(a \cdot A) = a^n \cdot \det(A)$

   scalar matrix $n \times n$
3. Addition, scalar multiplication etc. — intuitive (see p. 216)
4. Let $\vec{C_1}, \vec{C_2}, \dots \vec{C_r}$ be $n \times 1$-columns and B be a $n \times n$ matrix.
   Let
   $$\underset{n \times r}{C} = [\vec{C_1}, \vec{C_2}, \dots \vec{C_r}].$$
   Then
   $$BC = [B\vec{C_1}, B\vec{C_2}, \dots B\vec{C_r}]$$
   (i.e., matrix B multiplies each column of matrix C).
5. VERY IMPORTANT PROPERTY:

Let $A = [\vec{A_1}, \vec{A_2}, \dots \vec{A_n}]$.

$n \times n$ matrix $\quad$ $n \times 1$ vectors.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$\boxed{\begin{aligned} A\vec{x} &\equiv [\vec{A_1}, \vec{A_2}, \vec{A_3}, \ldots \vec{A_n}]\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1\vec{A_1} + x_2\vec{A_2} + \ldots + x_n\vec{A_n} \end{aligned}} \quad (6)$$

Ex. 2 Verify (6) for $n=2$.

$$\begin{pmatrix} \underbrace{\begin{matrix} a_{11} \\ a_{21} \end{matrix}}_{\vec{A_1}} & \underbrace{\begin{matrix} a_{12} \\ a_{22} \end{matrix}}_{\vec{A_2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}\cdot x_1 + a_{22}x_2 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}_{\vec{A_1}} + x_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}_{\vec{A_2}}$$

✓

$\boxed{2b}$ Calculus.

1. Derivatives — just differentiate each entry:

$$A' \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}' = \begin{pmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{pmatrix}.$$

Note: This should not be confused with the derivative of a determinant (see p. 20-5, Fact 4).

2. Derivatives of sum and product — just as for scalars (p. 219).

3. Integration – just integrate each term.

Ex. 3 Find $\int A(t)dt$, $A(t)=\begin{pmatrix} e^{3t} & 2t \\ 0 & -1 \end{pmatrix}$.

Sol'n: $\int A(t)dt=\begin{pmatrix} \int e^{3t}dt & \int 2t\,dt \\ \int 0\,dt & \int (-1)dt \end{pmatrix}$

$$=\begin{pmatrix} \frac{1}{3}e^{3t}+C_{11} & t^2+C_{12} \\ 0+C_{21} & -t+C_{22} \end{pmatrix}=\begin{pmatrix} \frac{1}{3}e^{3t} & t^2 \\ 0 & -t \end{pmatrix}+\underbrace{\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}}$$

the integration "constant" becomes a constant matrix

HW. Sec. 4.1. 1, 2, 4 ← simple operations
7 ← inversion of a 2×2
16, 17, 19, 21 ← integration
25 ← $(A^2)' \neq 2AA'$, Hint: Take $A=\begin{pmatrix} a(t) & b \\ 0 & c(t) \end{pmatrix}$ ← const

Answ. #4: $t^2(-t-1)$
#16: $\begin{pmatrix} -t+c_1 \\ t^2+c_2 \end{pmatrix}$

EC #1 #27 + 28 (about $(A^{-1})'$) ← together

#30 (+ read Ex. 1 on pp. 214-215).