Here we will prove the special case of Liouville's Theorem, called Abel's Thm; also, we'll do so only for $n=2$. The general case of Abel's Thm. is stated in Thm. 3.6.

Abel's Theorem for $n=2$

Let u, v be two solutions of a (DE-2):

$$y'' + p(t)y + q(t)y = 0. \tag{9a-1}$$

From Lec. 11, topic 2b, their Wronskian is:

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \tag{9a-2}$$

Then one has:

$$\boxed{W'(t) = -p(t)\,W(t)} \tag{9a-3}$$

or, equivalently:

$$\boxed{W(t) = W(t_0)\cdot \exp\left[-\int_{t_0}^{t} p(s)\,ds\right]} \tag{9a-4}$$

Proof:

Step 1: Rewrite (DE-2) (9a-1) as a system of first-order DEs.

By topic 0 of this Lecture, denote $y_1 = y$, $y_2 = y'$; then (9a-1) becomes (see Ex. 1 & 2):

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -py_2 - qy_1 \end{aligned} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{9a-5}$$

So, if u, v are two solutions of (9a-1), then

$$\vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v \\ v' \end{pmatrix}$$

are two solutions of the equivalent system (9a-5), and their Wronskian is, by Eq. (7) on p. 23-6:

$$W = \det \begin{vmatrix} u & v \\ u' & v' \end{vmatrix},$$

which is the same as the definition (9a-2) for (9a-1).

Now, the matrix P defined on p. (23-9), is:

$$P = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \Rightarrow tr[P] = 0 + (-p) = -p \qquad (9b\text{-}1)$$

Using (9b-1), one can easily verify that the Liouville Thm 4.4 (formula (8a) on p. 23-9) reduces to Abel's Thm. (9a-3).

Step 2 Since a Wronskian is a determinant, we need some auxiliary facts about determinants (stated below for $n=2$, but true in general):

$\boxed{\text{Fact 1}}$

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0$$

(if two rows or columns in a matrix are the same, det=0).

$\boxed{\text{Fact 2}}$

$$\begin{vmatrix} a_1 & a_2 \\ k b_1 & k b_2 \end{vmatrix} = k \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$\boxed{\text{Fact 3}}$

$$\begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$\boxed{\text{Fact "2+3"}}$

$$\begin{vmatrix} a_1 & a_2 \\ kb_1+mc_1 & kb_2+mc_2 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + m\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$\boxed{\text{Fact 4}}$ (product rule for determinants)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}' = \begin{vmatrix} a_1' & a_2' \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1' & b_2' \end{vmatrix}$$

(e.g., $(a_1 b_2)' = a_1' b_2 + a_1 b_2'$)

<u>Step 3</u> Proof of (9a-3)

Differentiate the Wronskian (9a-2):

$$\mathbf{W}' = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}' \overset{\boxed{F4}}{=} \underbrace{\begin{vmatrix} u' & v' \\ u' & v' \end{vmatrix}}_{=0 \text{ by } \boxed{F1}} + \begin{vmatrix} u & v \\ u'' & v'' \end{vmatrix}$$

$$\overset{\text{by the DE (9a-1)}}{=} \begin{vmatrix} u & v \\ (-pu'-qu) & (-pv'-qv) \end{vmatrix} \overset{\boxed{F\text{ "2+3"}}}{=}$$

$$= -p\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} - q \cdot \underbrace{\cancel{\begin{vmatrix} u & v \\ u & v \end{vmatrix}}}_{0 \text{ by } \boxed{F1}} = -\mathbf{p} \cdot \mathbf{W}.$$

This proves (9a-3).