# Lecture 27: Using LT to solve Initial Value Problems

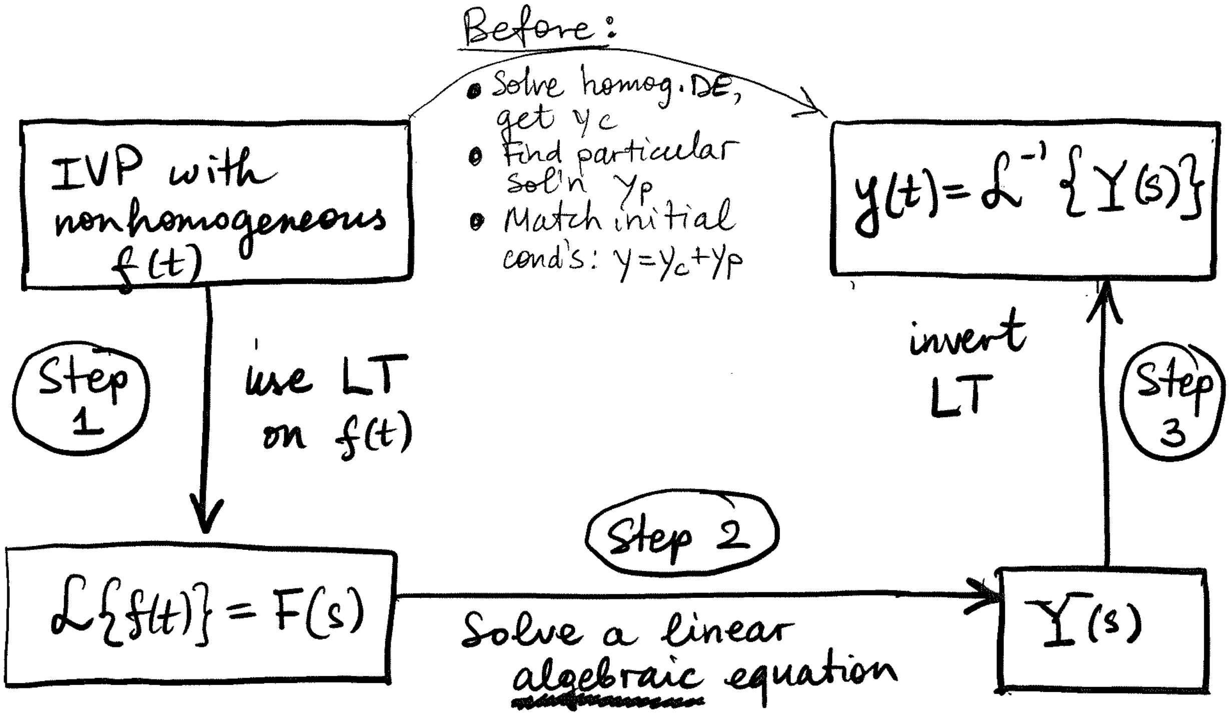
## ⓪ Goal of this lecture

Recall the goal announced in Lec. 26:
Instead of solving an IVP using DE-solving techniques studied in earlier Chapters, we will solve it in a round-about way.

IVP to be solved:

$$\begin{cases} \text{"homogeneous DE for } y(t)\text{"} = f(t) \\ \text{Initial Conditions for } y(t_0) \end{cases}$$

To be consistent with notations in the book, switch from "$g(t)$" to "$f(t)$" for the nonhomogeneous term.



to get: $$Y(s) = \underbrace{\Phi(s)}_{\text{from homogeneous DE}}\left(\underbrace{F(s)}_{\text{from } f(t)} + \underbrace{I(s)}_{\text{from Initial cond's}}\right)$$

So, to develop this <u>new method</u> of solving a nonhomogeneous DE, we will:

**For Step 1)** Continue learning about L.T.;

**For Step 2)** Learn how to find $\Phi(s)$ and $I(s)$ above;

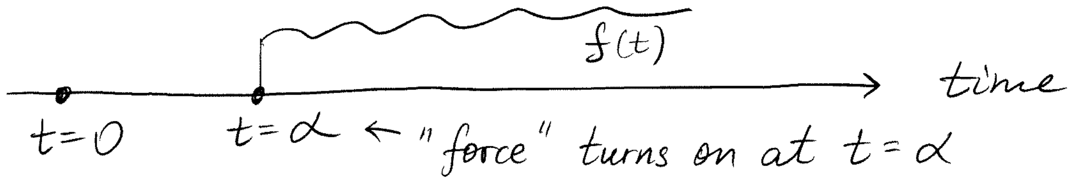
**For Step 3)** Review how to invert LT

(we already did one example of $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ in Ex. 2 of Lec. 26.)

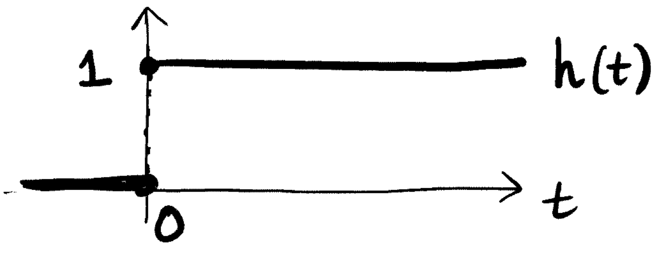
## ① For Step 1

### $\boxed{1a}$ The Heavyside step-function

In some applications, one has the nonhomogeneous terms that can begin acting <u>after</u> the initial time:



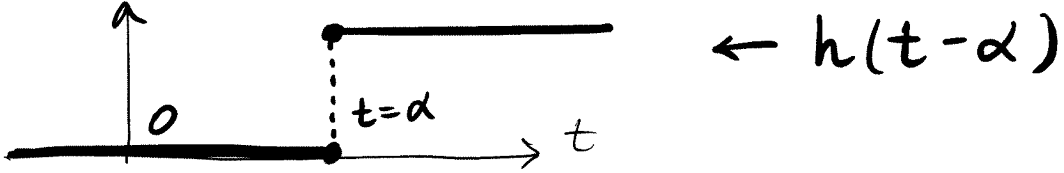
To describe this, use the Heavyside step-function:



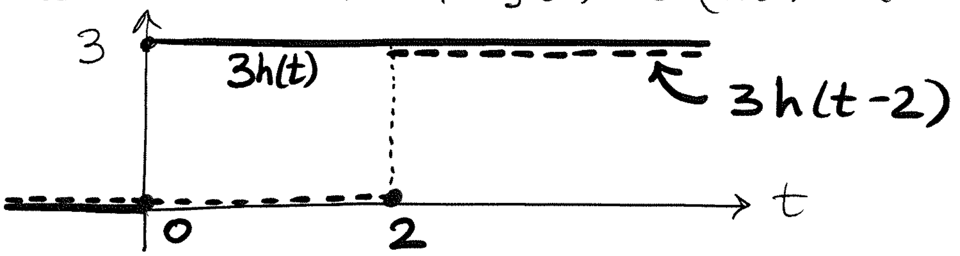
$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

**Note:** The value at a single point, $t=0$, is <u>not</u> important.

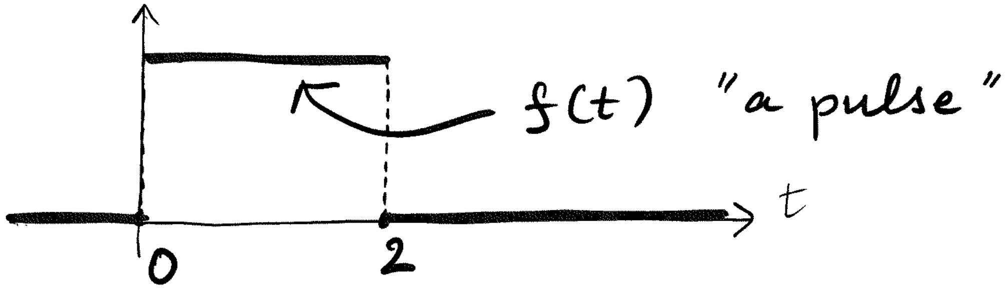
**Ex. 1(a)**



**Ex. 1(b)** Find $f(t) = 3\left(h(t) - h(t-2)\right)$



(shown as lower than "3" for better visibility)



Let us now find $\mathcal{L}\{h(t)\}$:

$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st} \cdot \underset{\uparrow\, h(t) \text{ for } t>0}{1}\, dt = \frac{e^{-st}}{-s}\Big|_0^\infty \underset{\uparrow}{=}$$

See Ex. 1/Lec.26 or Ex.2/Sec.5.1 for a more accurate treatment

$$= \frac{\cancel{e^{-\infty}}^{\,0} - \cancel{e^{-0}}^{\,1}}{-s} = \frac{1}{s}$$

Note 1: This appears to be the same answer as $\mathcal{L}\{1\}$ (see Ex. 1(a) in Lec. 26 (with $a=0$) or the Table on p. 340–341 in the book). This is expected, since LT only "cares about" $t>0$, where $h(t)=1$.

Note 2: The real utility of $h(t)$ is to describe processes that start at $t = \alpha > 0$, and so let's find $\mathcal{L}\{h(t-\alpha)\}$.

$$\mathcal{L}\{h(t-\alpha)\} = \int_0^\infty e^{-st} h(t-\alpha)\,dt \overset{\text{assume } \alpha>0}{=}$$

$$\underbrace{\int_0^\alpha e^{-st}\cdot 0\,dt}_{0} + \int_\alpha^\infty e^{-st}\cdot 1\cdot dt = \int_\alpha^\infty e^{-st}\,dt =$$

$$\underset{\substack{\uparrow\\ \text{trick to be repeatedly}\\ \text{used below}}}{=} \int_\alpha^\infty e^{-s[(t-\alpha)+\alpha]}\,dt = e^{-\alpha s}\int_\alpha^\infty e^{-s(t-\alpha)}\,dt$$

$u = t-\alpha$

$t: \alpha \to \infty$

$u: 0 \to \infty$

$du = dt$

$$= e^{-\alpha s}\int_0^\infty e^{-su}\,du \underset{\substack{\uparrow\\ \mathcal{L}\{h(u)\}}}{=} e^{-\alpha s}\cdot\frac{1}{s}$$

Thus:

$$\boxed{\mathcal{L}\{h(t-\alpha)\} = \frac{e^{-\alpha s}}{s}}$$

## $\boxed{1b}$ LT of $e^{at}$, $t^n$, $\sin\omega t$, $\cos\omega t$

The first one was done in Ex. 1(a) in Lec. 26; the rest are done on pp. 330-331 in the book. (Also, $\mathcal{L}\{t^1\}$ was done in Ex. 1(b) in Lec. 26.)

They are all done via <u>integration by parts</u>.

See Table on pp. 340-341; you will need LT of $h(t)$, $e^{at}$, $t^n$, sin, cos for HW & Final Exam, but you are <u>not</u> required to remember any of them.

**Must also see Ex. 1/Sec. 5.2** for some representative cases.

## 1c Shift Theorems

Under the usual conditions on $f(t)$ when all the LT's below exist (see Lec. 26, topic ④ = Thm. 5.1), one has:

**Theorem 5.4 (a)**

$$\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$$

Equivalently:

$$e^{\alpha t} f(t) = \mathcal{L}^{-1}\{F(s-\alpha)\}$$

**Theorem 5.4 (b)**

$$\mathcal{L}\{h(t-\alpha) f(t-\alpha)\} = e^{-\alpha s} F(s)$$

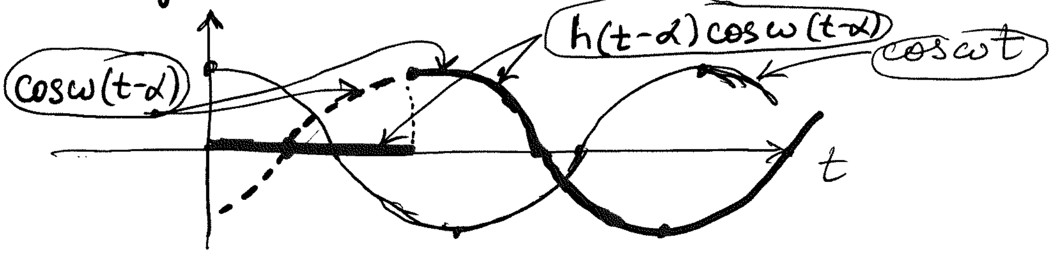
Equivalently:

$$h(t-\alpha) f(t-\alpha) = \mathcal{L}^{-1}\{e^{-\alpha s} F(s)\}$$

See proofs of both (a) & (b) in the book (optional); the proof of (b) is analogous to the proof in Note 2 on pp. 27-3, 27-4 above.

Note 1 = Clarification about the **difference between** $h(t-\alpha)f(t-\alpha)$ & $f(t-\alpha)$:

- $f(t-\alpha)$ is a shifted copy of $f(t)$; in general, $f(t-\alpha) \not\equiv 0$ for $0<t<\alpha$.
- $h(t-\alpha)f(t-\alpha)$ is a shifted copy of $f(t)$ **which begins at** $t=\alpha$; $h(t-\alpha)f(t-\alpha) \equiv 0$ for $0<t<\alpha$.



Note 2 = "do not invent your own formulas"

We know from Thm. 5.2 (Lec. 26, topic ⑤) that

$$\mathcal{L}\{\underbrace{c_1 f_1(t) + c_2}_{\text{const}} f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

But: $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

and similarly,

$$\mathcal{L}^{-1}\{F(s)G(s)\} \neq \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}$$

E.g., from Thm. 5.4(a):

$$\mathcal{L}\{e^{at}f(t)\} = F(s-\alpha) \neq \mathcal{L}\{e^{at}\}\mathcal{L}\{f(t)\} \left(= \tfrac{1}{s}\cdot F(s)\right)$$

A more complicated (and interesting!) relation between $\mathcal{L}\{fg\}$ & $\mathcal{L}\{f\}$, $\mathcal{L}\{g\}$ does exist, but we won't have the time to learn it (Sec. 5.6, Thm. 5.7).

Note 3: The Shift Theorems allow us to expand the Table on pp. 340-341; see entries 9-12 & 15 there and also Ex. 2/Sec. 5.2.

Ex. 2(a) Find $\mathcal{L}\{h(t-3)\cos 4(t-3)\}$.

$$\mathcal{L}\{h(t-3)\cos 4(t-3)\} \underset{\substack{\uparrow \\ \text{Thm. 5.4(b)}}}{=} e^{-3s}\cdot\mathcal{L}\{\cos 4t\} \underset{\substack{\uparrow \\ \text{Table}}}{=} e^{-3s}\cdot\frac{s}{s^2+4^2}$$

(Recall Note 1 above: $\mathcal{L}\{h(t-3)\cos 4(t-3)\} \neq \mathcal{L}\{\cos 4(t-3)\}$! The latter needs to be found from $\cos(4t-12) = \cos 4t\cos 12 + \sin 4t \sin 12$.)

**Ex. 2(b)** Find $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\}$

In $e^{-\alpha s}$, $\alpha = 1$; the expression looks like Thm. 5.4(b).

So: $\mathcal{L}^{-1}\left\{\frac{e^{-1\cdot s}}{s^2+4}\right\} = h(t-1)\,f(t-1)$, where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \underset{\substack{\uparrow \\ \text{Table}}}{=} \mathcal{L}^{-1}\left\{\boxed{\tfrac{1}{2}}\cdot\frac{\boxed{2}}{s^2+\boxed{2}^2}\right\} = \boxed{\tfrac{1}{2}}\sin \boxed{2}t$$

So, $f(t-1) = \frac{1}{2}\sin 2(t-1)$; Answer: $h(t-1)\cdot\frac{1}{2}\sin 2(t-1)$.

**Ex. 2(c)** Find $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-3)^2+4}\right\}$.

This $= h(t-1)\,f(t-1)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-\underset{\uparrow\alpha}{3})^2+4}\right\} =$

$$\underset{\substack{\uparrow \\ \text{Thm. 5.4(a)}}}{=} e^{3t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \underset{\substack{\uparrow \\ \text{Ex. 2(b)}}}{=} e^{3t}\cdot\frac{1}{2}\cdot\sin 2t.$$

Answer: $h(t-1)\cdot\frac{1}{2}e^{3(t-1)}\sin 2(t-1)$

**MUST also READ Ex. 2 in textbook (Sec. 5.2).**

## ② For Steps 2 & 3 (see p. 27-2)

To take LT of a DE, we need:

**Thm. 5.5** Assuming that all the LTs below exist:

(a) $\mathcal{L}\{y'(t)\} = s\cdot\mathcal{L}\{y(t)\} - y(0) \equiv s\cdot Y(s) - y(0)$

(b) $\mathcal{L}\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$

(will not need part (c) from the textbook).

Proof of (a): Integrate by parts

$$\mathcal{L}\{y'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{g}\,\underbrace{y'(t)}_{f'}\,dt =$$

$$\int_a^b g f' dt = gf\Big|_a^b - \int_a^b g' f\, dt$$

$$= e^{-st}\cdot y(t)\Big|_0^\infty - \int_0^\infty \underbrace{(-s)e^{-st}}_{g'}\cdot \underbrace{y(t)}_{f}\,dt$$

$$= \underset{0}{\cancel{e^{-\infty}}}\, y(\infty) - \underset{1}{\cancel{e^{-0}}}\, y(0) + s\underbrace{\int_0^\infty e^{-st} y(t)\, dt}_{\mathcal{L}\{y(t)\}}$$

$$\equiv sY(s) - y(0).$$

Proof of (b): $\mathcal{L}\{y''(t)\} = \mathcal{L}\{(y')'\} \underset{\uparrow}{=}$

See p. 334 (a 1-liner) ⟶ Apply part (a) twice.

Ex. 3(a) (= simplified Ex.3/book, so also See Ex.3/book)

Solve the IVP: $y' - 3y = 6 \xleftarrow{} f(t)$, $y(0) = 5$.

Sol'n: 1) $\mathcal{L}\{y' - 3y\} = \mathcal{L}\{6\} \underset{\uparrow}{=} 6\cdot \underbrace{\mathcal{L}\{h(t)\}}_{\uparrow}$

$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6\cdot\frac{1}{s} \xleftarrow{\quad}$ Table

See Note 1 after Ex.1

↓ Thm. 5.5(a)

$$(sY(s) - y(0)) - 3Y(s) = \frac{6}{s}$$

$$(s-3)Y(s) = \frac{6}{s} + \cancel{y(0)}^{\,5}$$

$$Y(s) = \underbrace{\frac{1}{s-3}}_{\Phi(s)}\cdot\Big(\underbrace{\frac{6}{s}}_{F(s)} + \underbrace{5}_{I(s)}\Big)$$ ⟵ See the diagram on p. 27-1.

2) We now need to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$\mathcal{L}^{-1}\{Y(s)\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\cdot\frac{6}{s}\right\}}_{\text{1st term}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-3}\cdot 5\right\}}_{\text{2nd term}}$$

1st term: Use **partial fractions** (see the Background Info sheet and HWs ## 5 & 7):

$$\frac{6}{s(s-3)} = \frac{A_1}{s} + \frac{A_2}{s-3}, \quad A_1 = -2,\ A_2 = 2$$

$$\mathcal{L}^{-1}\left\{-\frac{2}{s}+\frac{2}{s-3}\right\} = \underbrace{-2h(t)}_{\text{from Table}} + \underbrace{2h(t)e^{3t}}_{\text{using Thm. 5.4(a) and the previous term}}$$

2nd term:

$$\mathcal{L}^{-1}\left\{\frac{5}{s-3}\right\} \overset{\text{as above}}{=} 5h(t)e^{3t}.$$

Answer: $y(t) = -2h(t) + \underbrace{2h(t)e^{3t} + 5h(t)e^{3t}}_{7h(t)e^{3t}}$

for $t>0$, $h(t)=1$

$$\overset{\Downarrow}{=} -2 + 7e^{3t} \quad (t>0).$$

**Ex. 3(b)** Same, but for $f(t) = \begin{cases}0, & t<2\\ 6, & t\geq 2\end{cases} \equiv 6h(t-2)$.

Sol'n:

1) Proceed as in 1) of Ex. 3(a); the difference will be only in $\mathcal{L}\{f(t)\}$:

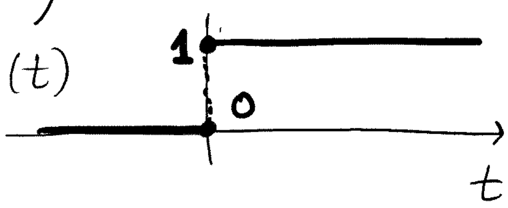
$$\mathcal{L}\{f(t)\} \underset{\uparrow}{=} 6\cdot\frac{e^{-\textcircled{2}s}}{s} \longleftarrow \text{came from } h(t-\textcircled{2})$$

Table (#14)
or Table (#1) + Thm. 5.4 (b)

2) $Y(s) = \frac{1}{s-3}\left(6\frac{e^{-2s}}{s} + 5\right)$

$$y(t) = \mathcal{L}^{-1}\Big\{\underbrace{\underbrace{e^{-2s}}_{\substack{\uparrow\\ \text{use}\\ \text{Thm. 5.4(b)}}}\cdot\underbrace{\frac{6}{s(s-3)}}_{\substack{\uparrow\\ \text{from Ex. 3(a)}}}}_{h(t-2)\left(-2h(t-2)+2h(t-2)e^{3(t-2)}\right)} + \underbrace{\frac{5}{s-3}}_{\substack{\text{same as in}\\ \text{Ex. 3(a)}}}\Big\}$$

Now use the fact: $(h(t))^2 = h(t)$



$\Rightarrow$

**Answer:** $y(t) = -2h(t-2)\left(1-e^{3(t-2)}\right) + 5e^{3t}h(t).$

**Ex. 4** Solve the IVP

$$y'' - y = e^{-2t}, \quad y(0) = y_0, \; y'(0) = y_0'.$$

**Sol'n:**

1) $\mathcal{L}\{y''-y\} = \mathcal{L}\{e^{-2t}\}$

$\downarrow$ Thm. 5.5 (b) $\qquad\qquad \downarrow$ Table

$$\left(\underline{s^2Y(s)} - \underline{sy_0 - y_0'}\right) - \underline{Y(s)} = \frac{1}{s+2}$$

$$(s^2-1)Y(s) = \frac{1}{s+2} + \underline{sy_0 + y_0'}$$

$$Y(s) = \underbrace{\frac{1}{s^2-1}}_{\Phi(s)}\cdot\left(\underbrace{\frac{1}{s+2}}_{F(s)} + \underbrace{sy_0+y_0'}_{I(s)}\right)$$

$\leftarrow$ See the diagram on p. 27-1

$$= \frac{1}{(s^2-1)(s+2)} + y_0\frac{s}{s^2-1} + y_0'\frac{1}{s^2-1}$$

Each of these terms is done by partial fraction expansion:

$Y(s) =$

$$\left(\frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2}\right) + y_0\left(\frac{1/2}{s-1} + \frac{1/2}{s+1}\right) + y_0'\left(\frac{1/2}{s-1} - \frac{1/2}{s+1}\right)$$

2) Use inverse LT (again, use the Table):

$$y(t) = \left(\frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}\right) + \frac{y_0}{2}\left(e^{t} + e^{-t}\right) + \frac{y_0'}{2}\left(e^{t} - e^{-t}\right).$$