Lecture 4 Some applications of linear DEs

① Population models (Sec. 2.4)

Simplest model:

$$\boxed{\text{Rate of population change}} = \boxed{\text{Rate of population increase}} - \boxed{\text{Rate of population decrease}}$$

$$\frac{dP}{dt} = r_b \cdot P - r_d \cdot P$$

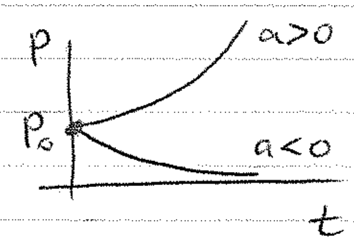
$r_b \cdot P$: births (assuming unlimited supply of resources & no competition)

$r_d \cdot P$: mortality

$$\frac{dP}{dt} = \underbrace{(r_b - r_d)}_{a = \text{const.}} P \qquad (1).$$

Solution is: $P(t) = P(0)\, e^{at}$

(Ex. 5 in Lec. 2).



See also Ex. 1, 2 in Sec. 2.4.

② Radioactive decay

Let $Q(t)$ be amount of radioactive material at time t.

It only decays ($r_f = 0$); let $r_d \equiv k$. Then

$$\frac{dQ}{dt} = -kQ$$

$$\Rightarrow \quad Q(t) = Q(0)\, e^{-kt}. \qquad (2)$$

Ex. 1 Suppose that 40% of some radioactive material decays in 4 years. Find the time that it takes 50% of the initial material to decay.

Sol'n: 1) $Q(t) = \underbrace{Q_0}_{Q(0)} \cdot e^{-kt}$.

We know $Q(4) = 0.6Q_0$, $\Rightarrow$

$$0.6 \cdot \cancel{Q_0} = \cancel{Q_0} \cdot e^{-k \cdot 4}$$

$$\ln 0.6 = -4k, \quad k = -\tfrac{1}{4} \ln 0.6 \simeq 0.128$$

2) If $Q(T) = 0.5Q_0$, $\Rightarrow$

$$0.5\, \cancel{Q_0} = \cancel{Q_0}\, e^{-kT}$$

$$\underbrace{\ln \tfrac{1}{2}}_{-\ln 2} = -kT \Rightarrow \boxed{T = \frac{\ln 2}{k}} \qquad (3)$$

half-life of material

For $k \simeq 0.128$, $T \simeq 5.4$ years.

<u>Radiocarbon dating</u>: This is a technique to measure the age of an archeological object based on the amount of $\underbrace{\text{Carbon-14}}_{{}^{14}_{6}C}$ in it.

<u>Some details of the technique:</u>

- ${}^{14}C$ is a radioactive isotope of carbon produced when the <u>stable</u> (non-decaying) isotope ${}^{12}C$ is exposed to cosmic radiation:

  " ${}^{12}C$ + radiation $\rightarrow$ ${}^{14}C$ "

- Once produced, ${}^{14}C$ has half-life of 5730 years.

- As the first approximation, one can assume that the ratio of ${}^{14}C$ to ${}^{12}C$ atoms in the atmosphere is known. It has the order of $1:10^{12}$ (one per trillion), although its value significantly varies depending on the location and the time moment back in the past. (See more on this in the Wikipedia article.)
  A look-up table of values of this ratio has been created over the years, based on some math models <u>and</u> on comparison of the radiocarbon dating results with results produced by other dating methods. A <u>lot</u> of work went into creation of ~~this~~ table.

- A living organism absorbs all carbon from the atmosphere and hence continues to maintain the same ratio of ^{14}C to ^{12}C as in the atmosphere throughout its lifetime. When the organism dies, the exchange of carbon with the atmosphere stops, and the amount of ^{14}C begins to decrease according to Eq. (2) (p. 4-2).
  Hence the ratio

$$\frac{Q_{14}(t)}{Q_{12}(t)} = \frac{Q_{14}(0)\, e^{-kt}}{Q_{12}(0)}$$

τ stays const; ^{12}C <u>not</u> radioactive

exponentially decays.

- A modern technique called accelerator mass spectrometry is capable of counting numbers of ^{12}C and ^{14}C and hence to determine the ratio $Q_{14}(t)/Q_{12}(t)$. (In 1950-1970s, researchers used different counting techniques.)
  Thus, knowing:
  $\frac{Q_{14}(t)}{Q_{12}(t)}$, $\frac{Q_{14}(0)}{Q_{12}(0)}$ (from look-up table), and k,

  one can find t — the age of the object.

This is further explored in some HW problems.

(3) Mixing problems (Sec. 2.3)

Ex. 2 The water in a shallow polluted lake initially contains 1 lb of mercury salts per 100,000 gallons of water.

Initially, there are 500,000 gallons of water in the lake.

Polluted water is pumped out of the lake at a rate 1000 gal/h and fresh water is pumped in at a rate 1500 gal/h.

What is the concentration of mercury salts in the lake when its volume reaches 600,000 gal?

Notations:

m = amount of mercury in lake;

W = amount of water

K = concentration of mercury.

(Obviously, $m = K \cdot W$.)

Given: $W(t=0) = 500{,}000$ (gal)

$m(t=0) = K(0) \cdot W(0) = \frac{1}{100{,}000} \cdot 500{,}000 = 5$ (lb)

flow of water in $\rightarrow F_{W,\,in} = 1500$ (gal/h)

flow of water out $\rightarrow F_{W,\,out} = 1000$ (gal/h)

$W(t_{final}) = 600{,}000$ (gal).

Find: $m(t_{final})$.

Assumptions (always hold in this Lecture): Mixing of polluted & fresh waters is instantaneous; thus concentration of mercury is uniform throughout lake.

Sol'n: 1) Balance of water:

$$\underbrace{\frac{dW}{dt}}_{\text{rate of change}} = \underbrace{F_{w,in}}_{\text{rate "in"}} - \underbrace{F_{w,out}}_{\text{rate "out"}}$$

$$\frac{dW}{dt} = 1500 - 1000 = 500 \quad \Rightarrow$$

$$W = 500t + C.$$

@t=0 $\quad W(0) = 500,000 \quad \Rightarrow \quad C = 500,000.$

$$W(t) = 500,000 + 500t. \qquad (4)$$

2) Balance of amount of mercury salts:

$$\frac{dm}{dt} = F_{m,in} - F_{m,out}$$

Note: Balance equation is always written for the amount, not concentration!

$$F_{m,in} = K_{in} \cdot F_{w,in} = 0 \cdot 1500 = 0$$

$$F_{w,out} = K_{out} \cdot F_{w,out} = \frac{m(t)}{W(t)} \cdot F_{w,out}$$

Thus,

$$\frac{dm}{dt} = -\frac{m}{W(t)} \cdot 1000 \qquad (5a)$$

$$m(0) = 5 \qquad (5b).$$

3) Solve IVP (5). It is of the form

$$y' + p(t)\, y = \cancel{g(t)} \to 0$$

$$\frac{1000}{W(t)} = \frac{1000}{500{,}000 + 500t} = \frac{1}{500 + 0.5t}$$

$$y(0) = \cancel{y_0} \to 5$$

and hence its solution is given by Eq. (9) of Lecture 2:

$$P(t) = \int_0^t \frac{dt_1}{500 + 0.5t_1} \Bigg|_{\substack{u = 500 + 0.5t_1 \\ du = 0.5\,dt_1}} = \int_{t_1=0}^{t_1=t} \frac{2\,du}{u}$$

$$= 2 \ln u \Big|_{t_1=0}^{t_1=t} = 2\left(\ln(500 + 0.5t) - \ln(500)\right).$$

m(t)

$$y(t) = y_0\, e^{-P(t)} = 5 \cdot e^{-2(\ln(500+0.5t) - \ln 500)}$$

$$= 5\, \frac{\left(e^{\ln 500}\right)^2}{\left(e^{\ln(500+0.5t)}\right)^2} = 5 \cdot \left(\frac{500}{500 + 0.5t}\right)^2$$

$$= \frac{5}{(1 + 0.001 \cdot t)^2}.$$

4) Find t_{final} & $m(t_{final})$.

From (4): $W(t_{final}) = 600{,}000 \Rightarrow$

$$600{,}000 = 500{,}000 + 500 \cdot t$$

$$t = 200 \text{ (hours)}$$

$$m(t_{final}) = \frac{5}{(1+0.001\cdot 200)^2} \approx 3.47 \text{ lb.}$$

<u>HW</u>: Sec. 2.4 : # 3 ← population
13 (a,c), 14, 15 ← radioactive

<u>Sec. 2.3</u> ## 2, 3, 1 ← $F_{in} = F_{out}$
7(a,c), 6 ← $F_{in} \neq F_{out}$.

Answer for 2: $50 \ln 10$.
Answer for 6: (a) 400 min
(b) 7.5 lb
(c) 10 lb at $t = 200$ min.