

(5-1)

Lecture 5: Solution of separable DEs.

(sec. 2.6)

Def: A DE of the form

$$\frac{dy}{dt} = - \frac{m(t)}{n(y)}$$

are called separable. The "-" on the r.h.s. is only to conform to the notations of the book.

To solve a separable DE:

$$\frac{dy}{dt} = - \frac{m(t)}{n(y)} \Rightarrow$$

$$n(y) dy = -m(t) dt$$

$$\int n(y) dy = -\int m(t) dt \Rightarrow$$

obtain an algebraic eqn. relating y and t .

Ex. 1 Linear separable DEs

$$\frac{dy}{dt} = -p(t) \cdot y \Rightarrow \frac{dy}{y} = -p(t) dt \Rightarrow$$

$$\ln y = -P(t) + C_1 \Rightarrow y = e^{-P(t) + C_1} = e^{-P(t)} \cdot e^{C_1}$$

Thus, we have proved Eq. (4) of Lec. 2:

$$y(t) = C e^{-P(t)}$$

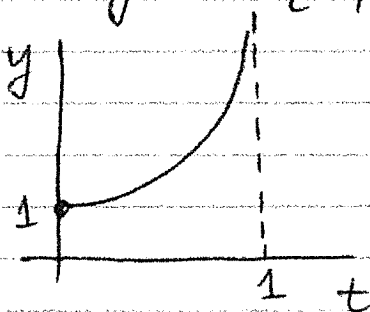
Ex. 2(a) $y' = y^2, y(0) = 1$

Sol'n: 1) $\frac{dy}{y^2} = 1 \cdot dt \Rightarrow -\frac{1}{y} = t + C$

$\Rightarrow y = \frac{-1}{t+C}$

2) $y(0) = 1 \Rightarrow 1 = \frac{-1}{0+C} \Rightarrow C = -1 \Rightarrow$

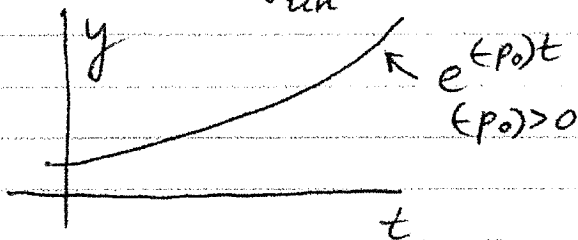
$y(t) = \frac{-1}{t-1} = \frac{1}{1-t}$



Note that the solution blows up at $t=1$, even though there are no discontinuous coeff's in the DE at all!

This should be contrasted with the solution of a linear equation, say,

$y_{lin} = e^{(p_0)t}, (p_0) > 0 :$



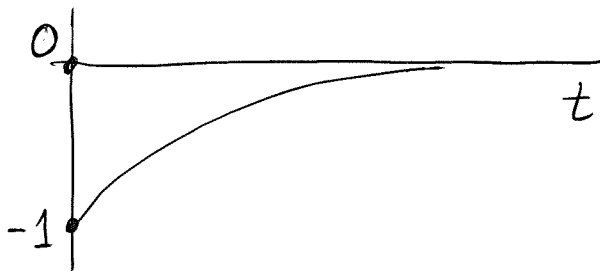
It grows exponentially, but exists for all t .

Ex. 2(b) $y' = y^2, y(0) = -1$

Sol'n: 1) $y = \frac{-1}{t+C}$ (same as in 2(a));

2) $-1 = \frac{-1}{0+C} \Rightarrow C = 1 \Rightarrow y = \frac{-1}{1+t}$

Note that this solution decays to 0:



Ex. 2(c) $y' = -y^2$, $y(0) = 1$

Sol'n: 1) $\frac{1}{y} = t + C$ (similarly to Ex. 2(a))

2) Use the initial condition to find C :

$$\frac{1}{1} = 0 + C \Rightarrow C = 1 \Rightarrow y = \frac{1}{t+1}$$

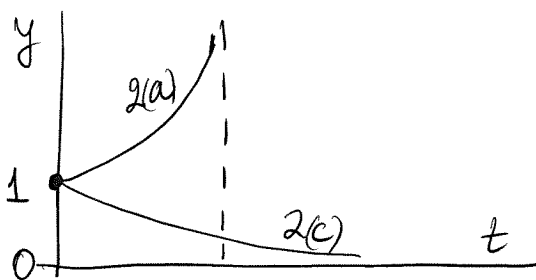
Again, this solution decays to 0.

Ex. 2(d) $y' = -y^2$, $y(0) = -1$

Similarly to Ex. 2(a), the solution blows up at $t=1$.

General Note #1 Qualitative behavior of the solution

Compare Ex. 2(a) & 2(c).



2(a): $y' = y^2 > 0 \Rightarrow y \uparrow$

2(c): $y' = -y^2 < 0 \Rightarrow y \downarrow$

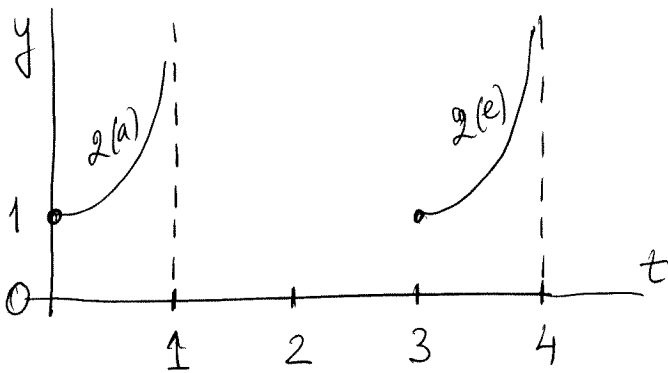
(Of course, without actually solving the DE, we cannot predict whether a blow-up will occur.)

Ex. 2(e): $y' = y^2, y(3) = 1$

1) $y = \frac{-1}{t+C}$ (same as in Ex. 2(a))

2) To find C: $1 = \frac{-1}{3+C} \Rightarrow C+3 = -1 \Rightarrow C = -1-3$

$y = \frac{-1}{t-\underbrace{1-3}_C} = \frac{-1}{(t-3)-1} = \frac{1}{1-(t-3)}$ (also = $\frac{1}{4-t}$).



Note that this is just a translated curve of 2(a).

This is the general case for autonomous DEs.

General note # 2

Consider two IVPs with the same autonomous DE

(I) $y' = f(y), y(0) = y_0$; let its sol'n be $y_I(t)$.

(II) $y' = f(y), y(t_0) = y_0$; let its sol'n be $y_{II}(t)$.

Claim:

$y_{II}(t) = y_I(t-t_0)$

" $y_I(t)$ shifted to the right by t_0 ."

Indeed: Rewrite (II) as:

• $\frac{dy}{d(t-t_0)} = f(y), y(t-t_0=0) = y_0$
 (since $dt = d(t-t_0)$, as $dt_0 = 0$)

• Denote $(t-t_0) = s$, a new variable

- Rewrite the IVP-II again:

$$\underbrace{\frac{dy}{ds} = f(y), \quad y(s=0) = y_0}_{y(0)}$$

But this is the same as IVP-(I), with t being replaced by $s = t - t_0$; hence indeed

$$y_{II}(t) = y_I(s) = y_I(t - t_0). \quad \checkmark$$

Ex. 2 (f) $y' = y^2, \quad y(0) = 0$

1) $y = \frac{-1}{t+C}$

2) Find C : $0 = \frac{-1}{0+C} \Rightarrow C = \infty, \Rightarrow$

$$y = \frac{-1}{t+\infty} = 0 \Rightarrow y = 0.$$

This could be guessed by simple inspection:
 $y=0$ satisfies: $y' = 0$ and $y^2 = 0$ and $y(0) = 0$.

Moral from Ex. 2 (f): For some nonlinear DEs with some special initial conditions, the solution can be found by simple inspection.

General Note # 3 Sometimes the algebraic eq. that can be obtained from

$$\int n(y) dy = - \int m(t) dt$$

cannot be solved for y . Then leave that algebraic equation in the form:

$$h(t, y) = C$$

↑ some nonlinear function.

Its solution $y(t)$ can be found (and plotted) on a computer. (See the result of Ex. 2, 3 of Sec. 2.6/book, but do not use their method, as it is too complicated.)

General Note # 4

Any autonomous DE $\frac{dy}{dt} = f(y)$

is separable : $\frac{dy}{f(y)} = dt$

$$(n(y) \equiv \frac{1}{f(y)}, m(t) \equiv -1)$$