# Lecture 7. Special cases when a nonlinear DE can and cannot be solved.

General idea:
Reduce the given DE to one of the two solvable forms — linear or separable.

## 1 Bernoulli eqs. (after Jacob Bernoulli, Swiss mathematician of XVII century).

General form:

$$y' + p(t)y = g(t)y^n$$

Ex. 1 $\quad y' + 2y = ty^4$ (compare with Ex. 1 of Sec. 2.5)

$$\underbrace{\frac{y'}{y^4}}_{\frac{1}{3}\left(-\frac{1}{y^3}\right)'} + 2\cdot\underbrace{\frac{y}{y^4}}_{\left(\frac{1}{y^3}\right)} = t$$

So denote

$$v = \frac{1}{y^3}$$

$$v' = -\frac{3y'}{y^4} \Rightarrow$$

$$\frac{y'}{y^4} = -\frac{1}{3}v'.$$

$$-\frac{1}{3}v' + 2v = t$$

$$v' - 6v = -3t \leftarrow \text{Linear nonhomogeneous DE for } v.$$

Then by Lec. 2,

$p(t) = -6$, $g(t) = -3t$, $P(t) = -6t$, $\Rightarrow$

$$v(t) = e^{6t}\big(\underbrace{\textstyle\int e^{-6t}\cdot(-3t)\,dt}_{\text{integrate by parts}} + C\big) =$$

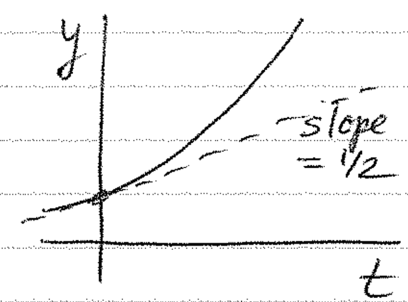
$$= e^{6t}\left(-\cancel{3}\cdot\left(-\frac{1}{\cancel{6}}\right)\left[t+\frac{1}{6}\right]e^{-6t} + C\right)$$

$$= \frac{1}{2}t + \frac{1}{12} + Ce^{6t}.$$

So $v = \frac{1}{y^3} \Rightarrow y = v^{-1/3} = \left(\frac{1}{2}t + \frac{1}{12} + Ce^{6t}\right)^{-1/3}$.

## (2) Using $\frac{dy}{dt} = 1/(dt/dy)$ to reduce the DE to linear

$$\frac{dy}{dt} = \frac{1}{-tp(y)+g(y)}$$




A solution is a curve $y = y(t)$; but can also be viewed as $t = t(y)$.

Now use $\frac{dy}{dt} = \frac{1}{(dt/dy)}$ (derivative of inverse function, Calc. I). Then

$$\frac{1}{dt/dy} = \frac{1}{-tp(y)+g(y)} \Rightarrow$$

new dependent variable

$$\frac{dt}{dy} = -p(y)t + g(y) \quad \leftarrow \text{linear DE for } t(y).$$

new independent variable

$$\frac{dt}{dy} + p(y)t = g(y)$$

Example @ home.

3. Use linear combination $z = at + by + c$.

$$y' = f(\underbrace{at + by + c}_{z}), \quad a, b, c = \text{const}.$$

$$z = (at + by + c) \Rightarrow y = \frac{z - at - c}{b},$$

$$\Rightarrow y' = \frac{z' - a}{b}.$$ Then the original DE becomes:

$$\frac{z' - a}{b} = f(z)$$ This is a separable DE.

$$z' = (b f(z) + a)$$

4. Reducible to separable; $z = \frac{y}{t}$.

$$\frac{dy}{dt} = f(\underbrace{\tfrac{y}{t}}_{z})$$

(product rule)

$$z = \frac{y}{t} \Rightarrow y = t \cdot z \Rightarrow y' = z + tz' \Rightarrow$$

DE: $z + tz' = f(z) \Rightarrow$

$$tz' = (f(z) - z) \Rightarrow z' = \frac{f(z) - z}{t}.$$

Separable DE.

5. Exact equations.

$\boxed{5a}$ Illustrate the idea with separable DEs.

$$\frac{dy}{dt} = -\frac{m(t)}{n(y)} \Rightarrow$$

$$n(y)\frac{dy}{dy} = -m(t) \quad \Rightarrow \underbrace{m(t)}_{\equiv M'(t)} + \underbrace{n(y)}_{N'(y)}\frac{dy}{dt} = 0 \quad (1)$$

$$\Rightarrow \frac{dM}{dt} + \underbrace{\frac{dN}{dy}\cdot\frac{dy}{dt}}_{} = 0 \Rightarrow \frac{d}{dt}(M(t) + N(y(t))) = 0 \quad (2)$$

Chain Rule: $\frac{dN(y(t))}{dt}$

$$\Rightarrow M(t) + N(y) = C \quad (3) \quad \Rightarrow \text{solve for } y(t).$$

$\boxed{5b}$ Generalize this idea.

Any DE can be written as

$$\frac{dy}{dt} = -\frac{m(t,y)}{n(t,y)} \quad \text{for some } m \,\&\, n.$$

Similarly to the above (Eq. (1)), this can be written as

$$m(t,y) + n(t,y)\frac{dy}{dt} = 0 \qquad (4)$$

Q1: For what $m$ & $n$ can this be written as

$$\frac{d}{dt}H(t, y(t)) = 0 \quad ? \qquad (5)$$

To answer this Question, we compute the l.h.s. of (5) using the <u>Calc. III Chain Rule.</u>

$$\begin{tikzcd} & H(t,y) \arrow[dl] \arrow[dr] & \\ s=t \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial s}\overbrace{\cancel{\frac{ds}{dt}}}^{1 \text{ (since } s=t)} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial y}\frac{dy}{dt}.$$

<u>Ex. 2</u> Let $H = t^2 + y^3$, $y = \sin t$.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial y}\frac{dy}{dt} = 2t + 3y^2 \cdot \cos t$$

Thus, an <u>Answer</u> to the Question is:

If
$$\underbrace{m(t,y)}_{\frac{\partial H}{\partial t}} + \underbrace{n(t,y)}_{\frac{\partial H}{\partial y}}\frac{dy}{dt} = 0 \quad \text{(4) (repeated)}$$

for some $H(t,y)$, then (4) becomes

$$\frac{d}{dt}H(t,y) = 0. \quad \text{(5) (repeated)}$$

<u>Q2:</u> So now when can we say of given $m, n$ that $m = \frac{\partial H}{\partial t}$, $n = \frac{\partial H}{\partial y}$ for some $H$?

A: By a Thm. from Calc. III,

$$\frac{\partial}{\partial y}\left(\frac{\partial H}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial y}\right)$$

when these derivatives are continuous. Then, if

$$\frac{\partial}{\partial y}\left(m(t,y)\right) = \frac{\partial}{\partial t}\left(n(t,y)\right), \qquad (6)$$

then (4) can be transformed to (5).

MUST READ PP. 66, 67 ON YOUR OWN

Details of this check and of finding $H$ (if (6) holds) are in Ex. 1, 2 of Sec. 2.7. Also, you saw the same calculation in Calc. III when you needed to check if some vector function $\vec{F} = \langle P, Q\rangle$ was a gradient:

$$\underset{\langle P,Q\rangle}{\vec{F}} \overset{?}{=} \underset{\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle}{\vec{\nabla} f}$$

and if so, then what is $f$?

$\boxed{5c}$ Caveat with this method

Suppose $\frac{\partial m}{\partial y} \neq \frac{\partial n}{\partial t}$, so

$$m + n\frac{dy}{dt} = 0$$

is not exact.

However, it may be possible to find

some function $\mu(t,y)$ such that

$$\underbrace{[\mu(t,y)\, m(t,y)]}_{m_1(t,y)} + \underbrace{[\mu(t,y)\, n(t,y)]}_{n_1(t,y)} \frac{dy}{dt} = 0 \tag{7}$$

is exact in the sense that

$$\frac{\partial m_1}{\partial y} = \frac{\partial n_1}{\partial t}. \tag{8}$$

Examples are in ##23–28 in Sec. 2.7.

The main problem, and the drawback of the method, is that finding $\mu(t,y)$ requires guessing. This is why we won't consider this technique (i.e. exact eqs.) in detail.

Yet, you still need to know what the term "exact eqs" means.

6) Riccati equation

This is the kind of the DE which cannot be solved analytically.

We study it here because it comes up in some applications (we'll see it later one more time). Also, it looks so deceptively simple that one may be tempted to try to solve it; so it will be useful to know that all such attempts will be futile.

Bernoulli: $y' + p(t)y = g(t)y^n$
(solvable)

Riccati: $y' + p(t)y = g(t)y^2 + h(t)$

Just adding this term makes this modification of the Bernoulli eq. <u>unsolvable</u>.

(It can still be solvable for some very special choices of $p(t)$, $g(t)$, $h(t)$.)

<u>HW</u>: <u>Sec. 2.5</u>: 13(a), 14(a), 17(a), 19(a);

$y' = \frac{1}{t+e^y}$, $y' = \frac{y}{y^2-t}$;

<u>Sec. 2.6</u> 35, 37, 36
38, 33, $y' = (x^2-y^2)/(3xy)$

<u>Sec. 2.7</u>: 11, 15, 1, 5, 7, 9, 23.