Lecture 9 Euler's method

Goal: Solve $\frac{dy}{dt} = f(t,y)$ approximately when it cannot be solved exactly.

1. **Idea:** Approximate dy/dt:

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h} \quad \text{for small } h.$$

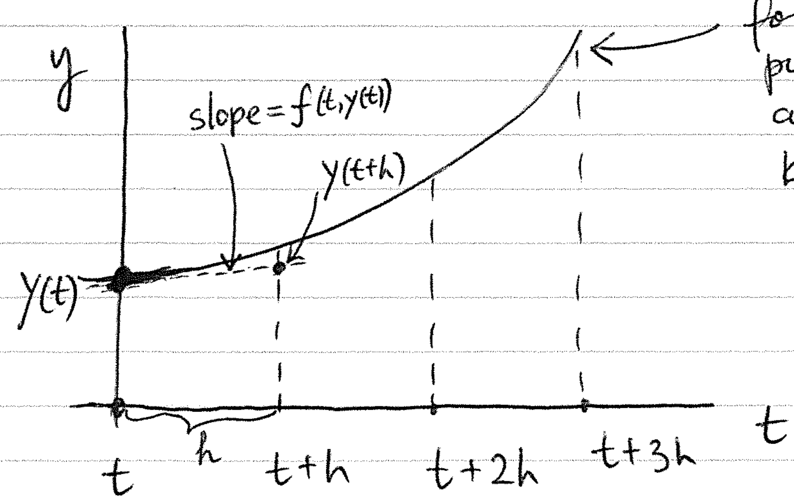
So, replace the DE with:

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t))$$

and solve for the "new" value $y(t+h)$ using the known "old" value $y(t)$:

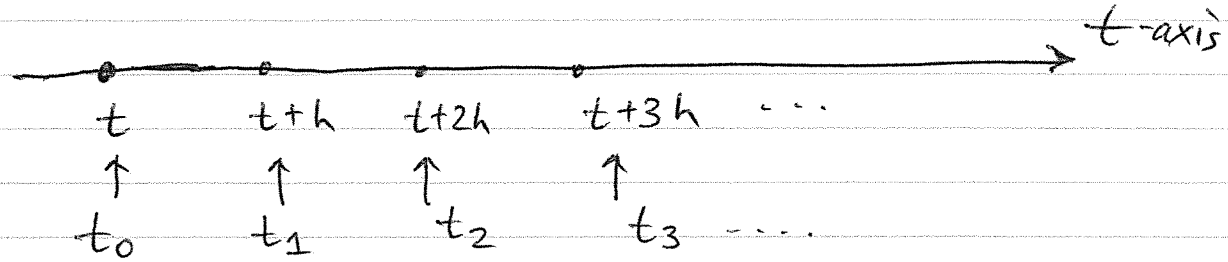
$$\boxed{y(t+h) = y(t) + h \cdot f(t, y(t))} \quad \text{Euler's Method.}$$

2. **Graphic representation**



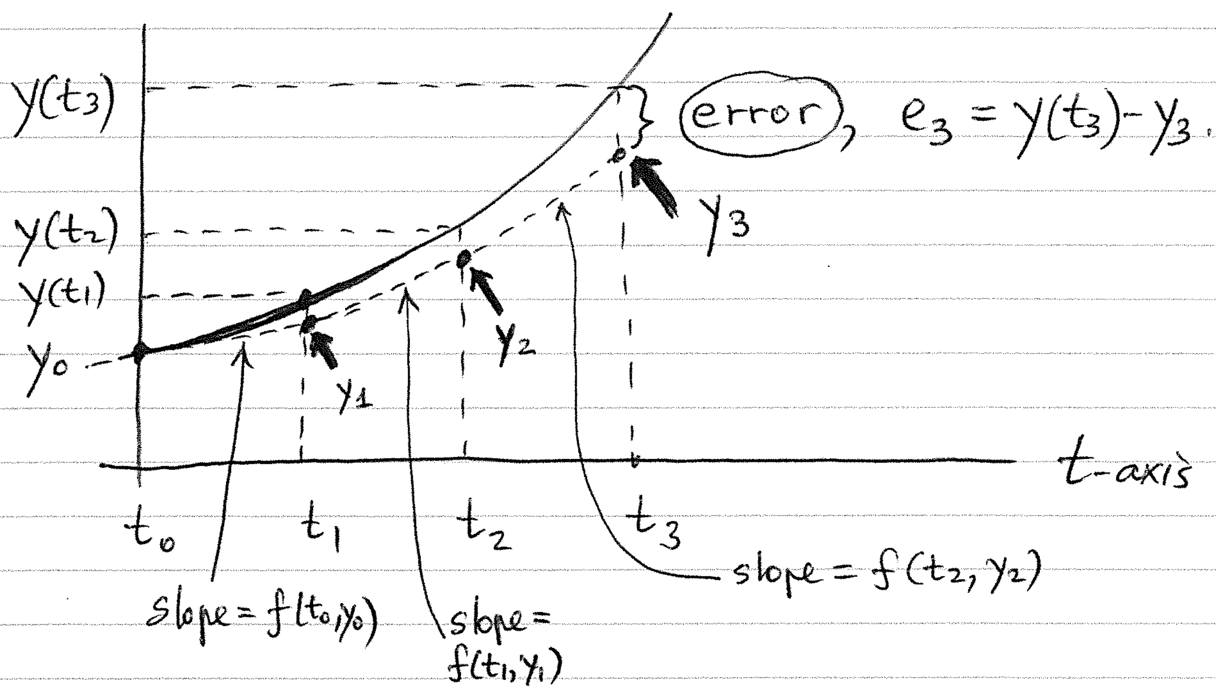
for this illustration purpose, let's assume that we know the sol'n $y(t)$

Let's introduce a more convenient notation:



$$t_k = t_0 + k \cdot h, \quad k = 0, 1, 2, \ldots$$

Then we can redraw the previous figure:



Another notation: write Euler's method as

$$\boxed{y_{k+1} = y_k + h \cdot f(t_k, y_k)}$$

y_k = approximate solution at $t = t_k$.

$y(t_k)$ = exact (and usually unknown) solution at $t = t_k$

<u>Ex. 1</u> Compute y_1, y_2, y_3 for

$$y' = 0.5y + t, \quad y(t_0) = 3; \; t_0 = 0;$$

while using $h = 0.1$.

<u>Sol'n:</u> $$y_{\underbrace{k+1}_{=3}} = y_k + h \cdot f(t_k, y_k) \quad \Rightarrow \quad k = 2.$$

Thus need to do the calculations for $k = 0, 1, 2$.

<u>$k=0$</u>

$$y_{\underbrace{0+1}_{1}} = y_0 + h \cdot f(t_0, y_0)$$

$$= 3 + 0.1 \cdot \left(0.5 \cdot \overset{3}{\cancel{y_0}} + \overset{0}{\cancel{t_0}}\right)$$

$$= 3 + 0.1 \cdot (0.5 \cdot 3 + 0) = \overset{y_1}{3.15}$$

<u>$k=1$</u>

$$y_{\underbrace{1+1}_{2}} = y_1 + h \cdot f(t_1, y_1)$$

$$= 3.15 + 0.1 \cdot \left(0.5 \overset{3.15}{\cancel{y_1}} + \overset{0.1}{\cancel{t_1}}\right)$$

$$= 3.15 + 0.1 \cdot 1.675 = \overset{y_2}{3.3175}$$

<u>$k=2$</u> $$y_{\underbrace{2+1}_{3}} = y_2 + h \cdot f(t_2, y_2)$$

$$= 3.3175 + 0.1 \cdot \left(0.5 \cdot \overset{3.3175}{\cancel{y_2}} + \overset{0.2}{\cancel{t_2}}\right)$$

$$= 3.3175 + 0.1 \cdot 1.85875 = \overset{y_3}{3.503375}$$

Exact sol'n (Lec. 2): $y(\underset{0.3}{t_3}) = 3.53284$.

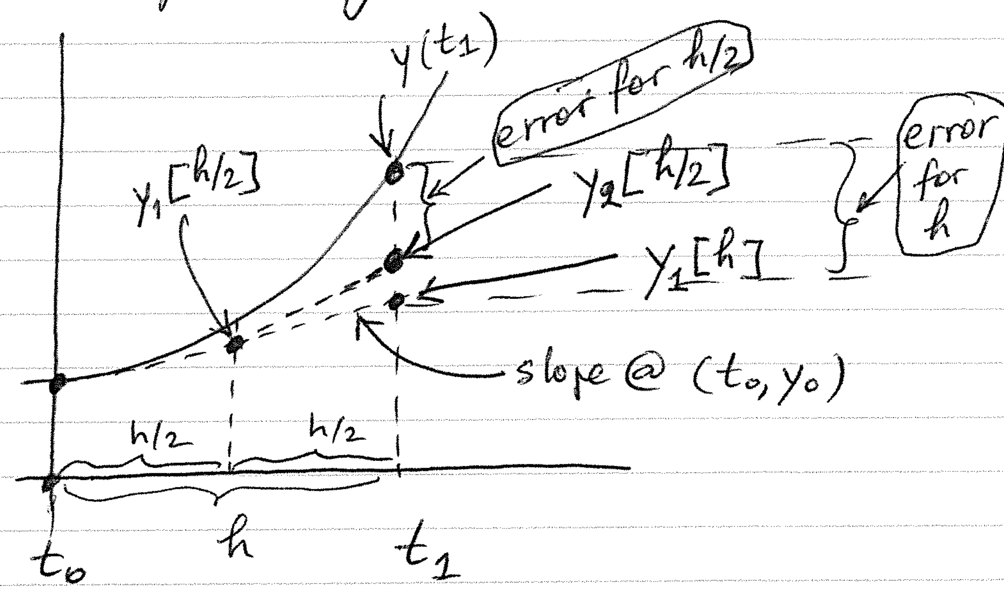
The error at $t_3 = 0.3$ is

$$|y(t_3) - y_3| = |3.53284 - 3.503375| \approx 0.0295,$$

or $\lesssim 0.9\%$.

Note: The error will decrease if we take a smaller h (expectedly)!

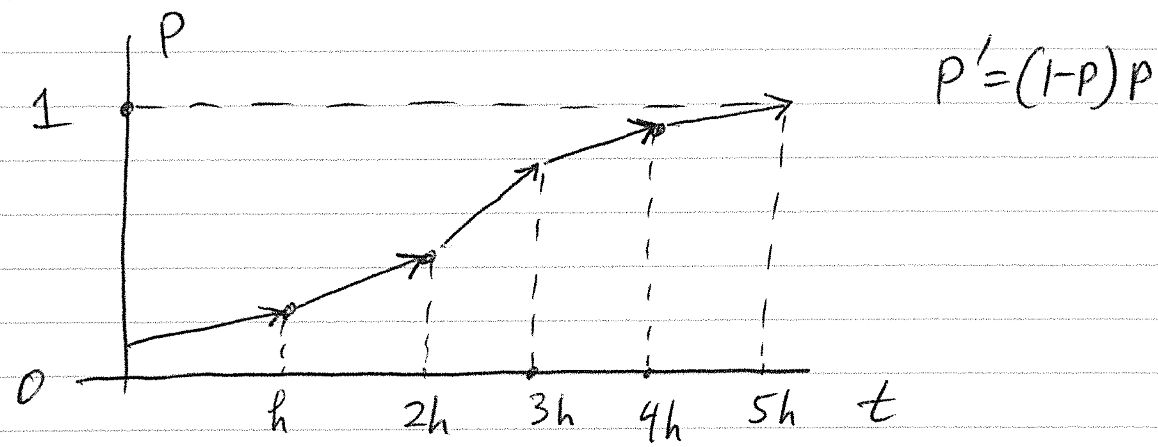
Why? Graphically:



We see that $y_2[h/2]$ is a better approximation to $y(t_1)$ than $y_1[h]$.

Mnemonically: The more accurate solution $y_{2k}[h/2]$ is always found between the exact sol'n $y(t_k)$ and the less accurate sol'n $y_k[h]$.

3 Relation to direction field:



The approximate solution found by Euler's method follows arrows of the direction field.

HW: Sec. 2.10 ## 3, 5.

Note for both: 1) Do not compute y_3 & e_3.

2) In addition to the assigned calculation, repeat it for $h = 0.05$ and compute y_1, y_2, y_3, y_4 for this new h, as well as e_1, e_2, e_3, e_4.

Verify that $e_4[h=0.05] \approx \frac{1}{2} e_2[h=0.1]$.