ADDING INTEGERS Mathematical folklore has it that Gauss discovered the formula $1+2+3+\cdots+n = n(n+1)/2$ when he was only ten years old. To occupy time, his teacher asked the students to add the integers from 1 to 100. Gauss immediately wrote an answer and turned his slate over. To his teacher's amazement, Gauss had the only correct answer in the class. Young Gauss had recognized that the numbers could be put in 50 sets of pairs such that the sum of each pair was 101:

$$(50+51)+(49+52)+(48+53)+\cdots+(1+100) = 50(101) = 5050.$$

Soon his brilliance was brought to the attention of the Duke of Brunswick, who thereafter sponsored the education of Gauss.

1.2 EXERCISES

Consider the matrices in Exercises 1–10.

a) Either state that the matrix is in echelon form or use elementary row operations to transform it to echelon form.

b) If the matrix is in echelon form, transform it to reduced echelon form.

1. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 & 2 & 3 \\ 2 & 0 & 1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 4 & -3 & 4 & 6 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

In Exercises 11–21, each of the given matrices represents the augmented matrix for a system of linear equations. In each exercise, display the solution set or state that the

11. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$

In Exercises 22–35, solve the system by transforming the augmented matrix to reduced echelon form.

22. $\begin{aligned} 2x_1 - 3x_2 &= 5 \\ -4x_1 + 6x_2 &= -10 \end{aligned}$

23. $x_1 - 2x_2 = 3$

24. $\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 4x_3 &= -3 \end{aligned}$

25. $\begin{aligned} x_1 + x_2 &= 2 \\ 3x_1 + 3x_2 &= 6 \end{aligned}$

26. $\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ 2x_1 - 2x_2 + 3x_3 &= 2 \end{aligned}$

27. $\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ -3x_1 - 3x_2 + 3x_3 &= -6 \end{aligned}$

28. $\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 3 \\ x_1 - 2x_2 - 2x_3 &= -2 \\ -x_1 + 16x_2 + 2x_3 &= 16 \end{aligned}$

29. $\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 7x_3 &= 8 \\ -x_1 + x_2 - 5x_3 &= -5 \end{aligned}$

30. $\begin{aligned} x_1 + x_2 \qquad\qquad - x_5 &= 1 \\ x_2 + 2x_3 + x_4 + 3x_5 &= 1 \\ x_1 \qquad - x_3 + x_4 + x_5 &= 0 \end{aligned}$

31. $\begin{aligned} x_1 \qquad + x_3 + x_4 - 2x_5 &= 1 \\ 2x_1 + x_2 + 3x_3 - x_4 + x_5 &= 0 \\ 3x_1 - x_2 + 4x_3 + x_4 + x_5 &= 1 \end{aligned}$

32. $\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ 2x_1 + x_2 &= 3 \end{aligned}$

33. $\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ 2x_1 + x_2 &= 2 \end{aligned}$

34. $\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 2 \\ -x_1 - 2x_2 &= -1 \end{aligned}$

35. $\begin{aligned} x_1 - x_2 - x_3 &= 1 \\ x_1 \qquad + x_3 &= 2 \\ x_2 + 2x_3 &= 3 \end{aligned}$

In Exercises 36–40, find all values a for which the system has no solution.

36. $\begin{aligned} x_1 + 2x_2 &= -3 \\ ax_1 - 2x_2 &= 5 \end{aligned}$

37. $\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 + 6x_2 &= a \end{aligned}$

38. $\begin{aligned} 2x_1 + 4x_2 &= a \\ 3x_1 + 6x_2 &= 5 \end{aligned}$

39. $\begin{aligned} 3x_1 + ax_2 &= 3 \\ ax_1 + 3x_2 &= 5 \end{aligned}$

40. $\begin{aligned} x_1 + ax_2 &= 6 \\ ax_1 + 2ax_2 &= 4 \end{aligned}$

In Exercises 41 and 42, find all values α and β where $0 \le \alpha \le 2\pi$ and $0 \le \beta \le 2\pi$.

41. $\begin{aligned} 2\cos\alpha + 4\sin\beta &= 3 \\ 3\cos\alpha - 5\sin\beta &= -1 \end{aligned}$

42. $\begin{aligned} 2\cos^2\alpha - \sin^2\beta &= 1 \\ 12\cos^2\alpha + 8\sin^2\beta &= 13 \end{aligned}$

43. Describe the solution set of the following system in terms of x_3:

$$x_1 + x_2 + x_3 = 3$$

a) Find the maximum value of x_3 such that $x_1 \ge 0$ and $x_2 \ge 0$.

b) Find the maximum value of $y = 2x_1 - 4x_2 + x_3$ subject to $x_1 \ge 0$ and $x_2 \ge 0$.

c) Find the minimum value of $y = (x_1 - 1)^2 + (x_2 + 3)^2 + (x_3 + 1)^2$ with no restriction on x_1 or x_2. [*Hint:* Regard y as a function of x_3 and set the derivative equal to 0; then apply the second-derivative test to verify that you have found a minimum.]

44. Let A and I be as follows:

$$A = \begin{bmatrix} 1 & d \\ c & b \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Prove that if $b - cd \ne 0$, then A is row equivalent to I.

45. As in Fig. 1.4, display all the possible configurations for a (2×3) matrix that is in echelon form. [*Hint:* There are seven such configurations. Consider the various positions that can be occupied by one, two, or none of the symbols.]

46. Repeat Exercise 45 for a (3×2) matrix, for a (3×3) matrix, and for a (3×4) matrix.

47. Consider the matrices B and C:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

By Exercise 44, B and C are both row equivalent to matrix I in Exercise 44. Determine elementary row operations that demonstrate that B is row equivalent to C.

48. Repeat Exercise 47 for the matrices

$$B = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

49. A certain three-digit number N equals fifteen times the sum of its digits. If its digits are reversed, the resulting number exceeds N by 396. The one's digit is one larger than the sum of the other two. Give a linear system of three equations whose three unknowns are the digits of N. Solve the system and find N.

50. Find the equation of the parabola, $y = ax^2 + bx + c$, that passes through the points $(-1, 6)$, $(1, 4)$, and $(2, 9)$. [*Hint:* For each point, give a linear equation

understanding that the loser gives each winner an amount equal to what the winner already has. After three games, each has lost just once and each has $24. With how much money did each begin?

52. Find three numbers whose sum is 34 when the sum of the first and second is 7, and the sum of the second and third is 22.

53. A zoo charges $6 for adults, $3 for students, and $.50 for children. One morning 79 people enter and pay a total of $207. Determine the possible numbers of adults, students, and children.

54. Find a cubic polynomial, $p(x) = a+bx+cx^2+dx^3$, such that $p(1) = 5$, $p'(1) = 5$, $p(2) = 17$, and $p'(2) = 21$.

In Exercises 55–58, use Eq. (2) to find the formula for the sum. If available, use linear algebra software for Exercises 57 and 58.

55. $1 + 2 + 3 + \cdots + n$

56. $1^2 + 2^2 + 3^2 + \cdots + n^2$

57. $1^4 + 2^4 + 3^4 + \cdots + n^4$

58. $1^5 + 2^5 + 3^5 + \cdots + n^5$

1.3 CONSISTENT SYSTEMS OF LINEAR EQUATIONS

We saw in Section 1.1 that a system of linear equations may have a unique solution, infinitely many solutions, or no solution. In this section and in later sections, it will be shown that with certain added bits of information we can, without solving the system, either eliminate one of the three possible outcomes or determine precisely what the outcome will be. This will be important later when situations will arise in which we are not interested in obtaining a specific solution, but we need to know only how many solutions there are.

To illustrate, consider the general (2×3) linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2. \end{aligned}$$

Geometrically, the system is represented by two planes, and a solution corresponds to a point in the intersection of the planes. The two planes may be parallel, they may be coincident (the same plane), or they may intersect in a line. Thus the system is either inconsistent or has infinitely many solutions; the existence of a unique solution is impossible.

Solution Possibilities for a Consistent Linear System

We begin our analysis by considering the $(m \times n)$ system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots \;\; & \;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

Our goal is to deduce as much information as possible about the solution set of system (1) without actually solving the system.

To that end, let $[A \mid \mathbf{b}]$ denote the augmented matrix for system (1). We know we can use row operations to transform the $[m \times (n+1)]$ matrix $[A \mid \mathbf{b}]$ ...

deduce the various possibilities for the solution set of (1), we will focus on the simple problem of analyzing the solution possibilities for the *equivalent* system represented b the matrix $[C \mid \mathbf{d}]$.

We begin by making four remarks about an $[m \times (n+1)]$ matrix $[C \mid \mathbf{d}]$ that is i reduced echelon form. Our first remark recalls an observation made in Section 1.2.

Remark 1: The system represented by the matrix $[C \mid \mathbf{d}]$ is inconsistent if and only i $[C \mid \mathbf{d}]$ has a row of the form $[0, 0, 0, \dots, 0, 1]$.

Our second remark also follows because $[C \mid \mathbf{d}]$ is in reduced echelon form. I particular, we know every nonzero row of $[C \mid \mathbf{d}]$ has a leading 1. We also know ther are no other nonzero entries in a column of $[C \mid \mathbf{d}]$ that contains a leading 1. Thus, if x_k is the variable corresponding to a leading 1, then x_k can be expressed in terms of othe variables that do not correspond to any leading ones in $[C \mid \mathbf{d}]$. Therefore, we obtain

Remark 2: Every variable corresponding to a leading 1 in $[C \mid \mathbf{d}]$ is a dependent vari able. (That is, each "leading-one variable" can be expressed in terms of the independen or "nonleading-one variables.")

We illustrate Remark 2 with the following example.

EXAMPLE 1 Consider the matrix $[C \mid \mathbf{d}]$ given by

$$[C \mid \mathbf{d}] = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[C \mid \mathbf{d}]$ is in reduced echelon form and represents the consistent system

$$\begin{aligned} x_1 + 2x_2 + \quad 3x_4 + \quad + 4x_6 &= 1 \\ x_3 + 2x_4 + \quad + 3x_6 &= 2 \\ x_5 + \; x_6 &= 2. \end{aligned}$$

The dependent variables (corresponding to the leading 1's) are x_1, x_3, and x_5. They can be expressed in terms of the other (independent) variables as follows:

$$\begin{aligned} x_1 &= 1 - 2x_2 - 3x_4 - 4x_6 \\ x_3 &= 2 \qquad\quad - 2x_4 - 3x_6 \\ x_5 &= 2 \qquad\qquad\qquad - x_6. \end{aligned}$$

Our third remark gives a bound on the number of nonzero rows in $[C \mid \mathbf{d}]$. Let r denote the number of nonzero rows in $[C \mid \mathbf{d}]$. (Later we will see that the number r is called the "rank" of C.) Since every nonzero row contains a leading 1, the number r is equal to the number of leading 1's. Because the matrix is in echelon form, there cannot be more leading 1's in $[C \mid \mathbf{d}]$ than there are columns. Since the matrix $[C \mid \mathbf{d}]$ has $n + 1$ columns, we conclude ...