Sec. 1.7 – PART 2

② Nonsingular & singular matrices

<u>Def</u>: An $n\times n$ matrix A is <u>nonsingular</u> if the only sol'n of $A\underline{x}=\underline{\theta}$ is $\underline{x}=\underline{\theta}$.

Equivalently, we have:

<u>Def*</u>: An $n\times n$ matrix A is <u>singular</u> if there is some $\underline{x}\neq\underline{\theta}$ such that $A\underline{x}=\underline{\theta}$.

Note: Singular is Special.

Relation with topic ①:

We can write $A=[\underline{A}_1, \underline{A}_2, \dots \underline{A}_n]$. Then:

$$A\underline{x}=\underline{\theta} \iff x_1\underline{A}_1+x_2\underline{A}_2+\dots+x_n\underline{A}_n=\underline{\theta}.$$

Also, $\underline{x}=\underline{\theta} \iff x_1=x_2=\dots=x_n=0$.

Then the above Def's are equivalent to:

<u>Thm. 12</u>

(A is nonsingular) $\iff$ (columns of A are lin. independent)

$n\times n$

<u>Thm. 12*</u>

(A is singular) $\iff$ (columns of A are lin. dependent)

<u>Ex. 5</u> For what a is matrix $A=\begin{pmatrix}1&2\\a&3\end{pmatrix}$ nonsingular?

<u>Sol'n</u>: By Ex. 3, $\underline{A}_1=\begin{pmatrix}1\\a\end{pmatrix}$ and $\underline{A}_2=\begin{pmatrix}2\\3\end{pmatrix}$ are lin. indep. for $a\neq 3/2$. Then by Thm. 12, A is nonsingular for $a\neq 3/2$.

Discussion: So, for $a=3/2$, A is singular. How does this conclusion jive with the definition of a singular matrix?

For $a=3/2$, $\underline{A_1} = \binom{1}{3/2}$, $\underline{A_2} = \binom{2}{3}$, $\Rightarrow \underline{A_2} = 2\underline{A_1}$.

Rewrite this as:

$$2\underline{A_1} - \underline{A_2} = \underline{\theta} \Leftrightarrow 2\cdot\underline{A_1} + (-1)\cdot\underline{A_2} = \underline{\theta}$$

Key Formula $\Rightarrow$

$$\underbrace{[\underline{A_1}, \underline{A_2}]}_{A}\binom{2}{-1} = \underline{\theta}$$

$$A \cdot \underline{x} = \underline{\theta}$$

So, the special $\underline{x}$ that makes $A\underline{x} = \underline{\theta}$ is $\underline{x} = \binom{2}{-1}$ (for this particular A).

Ex. 6 Prove that the identity matrix (e.g. for 3×3, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$) is nonsingular.

Discuss in class (emphasize that should **start with the Definition** of a sing. matrix, not lin. dep. of columns). Then computationally, it is the same proof as in our Ex. 2 (p. 75 in book). **See also Ex. 5 in book.**

Now we'll address the **BIG QUESTION**:
When does an $n\times n$ l.s. $A\underline{x} = \underline{b}$ have a unique sol'n?

Thm. 13 Let A be $n\times n$.

$$\boxed{(A\underline{x} = \underline{b} \text{ has a unique sol'n}) \Leftrightarrow (A = \text{nonsingular})}$$

We'll only prove the "$\Leftarrow$", and only for 2×2 ($n\times n$ is similar)

$$(A = \text{nonsingular}) \Rightarrow \left(A\underline{x} = \underline{b} : \begin{array}{l} \text{(a) has a sol'n } \underline{x}_0 \\ \text{(b) this } \underline{x} \text{ is unique} \end{array}\right)$$

Proof.

Given:

- $\underline{A}\underline{x} = \underline{\theta}$ only for $\underline{x} = \underline{\theta}$
- $x_1 \underline{A}_1 + x_2 \underline{A}_2 = \underline{\theta}$ only for $x_1 = x_2 = 0$.

Want:

(a) Some $\underline{x}$ exists that satisfies $A\underline{x} = \underline{b}$.

(b) This $\underline{x}$ is unique.

(a) Consider a set

$\{\underline{A}_1, \underline{A}_2, \underline{b}\}$: 3 vectors in $\mathbb{R}^2$, $\Rightarrow$ by Thm. 11 or Ex. 4, they are lin. dependent.

Then $c_1 \underline{A}_1 + c_2 \underline{A}_2 + c_3 \underline{b} = \underline{\theta}$, (☆)

where some of c_1, c_2, c_3 must be $\neq 0$. (☆☆)

Now we argue that for sure, $c_3 \neq 0$. Why?

Because if $c_3 = 0$, then (☆) reduces to

$c_1 \underline{A}_1 + c_2 \underline{A}_2 = \underline{\theta}$.

But by "Given", this is only possible for $c_1 = c_2 = 0$.

Then $c_3 = 0$ and $c_1 = c_2 = 0$, which contradicts (☆☆).

So, $c_3 \neq 0$, and we can divide (☆) by c_3:

$$\left(\frac{c_1}{c_3}\right) \underline{A}_1 + \left(\frac{c_2}{c_3}\right) \underline{A}_2 + \underline{b} = \underline{\theta}, \Rightarrow$$

$$\underbrace{\left(-\frac{c_1}{c_3}\right)}_{\text{some } x_1} \underline{A}_1 + \underbrace{\left(-\frac{c_2}{c_3}\right)}_{\text{some } x_2} \underline{A}_2 = \underline{b}$$

$$x_1 \underline{A}_1 + x_2 \underline{A}_2 = \underline{b} \Rightarrow \underbrace{[\underline{A}_1, \underline{A}_2]}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{some } \underline{x}} = \underline{b}$$

I.e., there indeed exists some $\underline{x}$ that satisfies $A\underline{x} = \underline{b}$.

(a) is proved

(b) Let's show that there are no other sol'ns.
By contradiction, suppose there are 2 sol'ns:
$A\underline{x} = \underline{b}$ and $A\underline{y} = \underline{b}$.
Subtract: $A\underline{x} - A\underline{y} = \underline{b} - \underline{b} \Rightarrow A\underbrace{(\underline{x}-\underline{y})}_{\underline{z}} = \underbrace{\underline{b}-\underline{b}}_{\underline{\theta}}$

$\Rightarrow A\underline{z} = \underline{\theta}$. But by "Given", this can be only when $\underline{z}=\underline{\theta} \Rightarrow \underline{x}-\underline{y} = \underline{\theta} \Rightarrow \underline{x} = \underline{y}$.

(b) is proved.

Proof of "$\Rightarrow$" in Thm. 13 (OPTIONAL):
($A\underline{x} = \underline{b}$ has a unique sol'n) $\Rightarrow$ (A = nonsingular).
Take $\underline{b} = \underline{\theta}$. Then $A\underline{x} = \underline{\theta}$ has a unique sol'n.
By inspection, $\underline{x} = \underline{\theta}$ is a sol'n, and now it is unique.
Then by Def of a nonsingular matrix, A=nonsingular.

Ex. 7. What can the number of solutions be for $\begin{pmatrix} 1 & 2 \\ a & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ c \end{pmatrix}$?

Sol'n: 1) Based on Thm. 13, we know that when the matrix is nonsingular, there is 1 sol'n.
From Ex. 5, matrix is nonsingular for $a \neq 3/2$.
So, $(a \neq 3/2) \Rightarrow$ (1 sol'n; regardless of c).

2) what if $a = 3/2$? Then the matrix is singular. Use Thm. 13 to explore the possible number of sol'ns.

Thm. 13 says:

$$(A\underline{x}=\underline{b} \text{ has 1 sol'n}) \Rightarrow (A = \text{nonsingular}).$$

Negate this statement and use laws of logic:

$$(A = \text{singular}) \Rightarrow (A\underline{x}=\underline{b} \text{ has not 1 sol'n})$$

means 0 or ∞

Now let's apply this to our example.

<u>$a = 3/2$</u>

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3/2 & 3 & c \end{array}\right) \xrightarrow[R_2 - \frac{3}{2}R_1]{} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & c-6 \end{array}\right)$$

- if $(c-6) \neq 0 \Rightarrow$ divide R_2 by $(c-6) \Rightarrow$

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 1 \end{array}\right) \Rightarrow \text{the l.s. in inconsistent (Sec. 1.3)}$$

$$\Rightarrow \boxed{0 \text{ sol'ns}}$$

- if $c - 6 = 0$ (i.e. $c = 6$):

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 = 4 \\ x_2 = \text{free} \end{array} \Rightarrow \boxed{\infty \text{ many sol'ns}}$$

Thus, a <u>complement</u> to Thm. 13:

$$\boxed{(A = \text{singular}) \Rightarrow \left(\begin{array}{l} A\underline{x} = \underline{b} \text{ can have either} \\ 0 \text{ or } \infty \text{ many} \\ \text{sol'ns} \end{array}\right)}$$

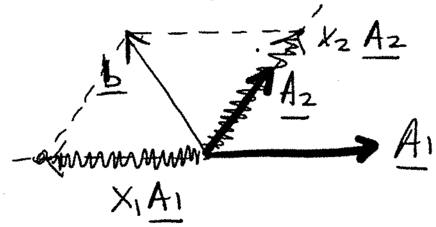
On the next page we illustrate this statement GEOMETRICALLY.

Geometric illustration of Thm. 13 and its complement:

number of sol'ns of $A\underline{x} = \underline{b}$.

$\underline{R^2}$, i.e. $A = [\underline{A_1}, \underline{A_2}]$.

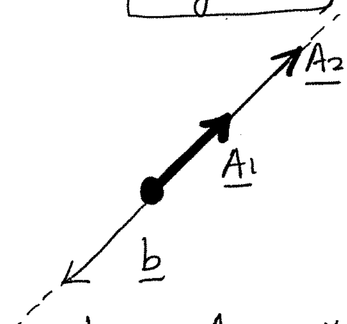
- A is $\boxed{\text{nonsingular}}$, $\underline{A_1}, \underline{A_2}$ lin. independent.



There is $\boxed{1 \text{ sol'n}}$ $\binom{x_1}{x_2}$ to

$x_1 \underline{A_1} + x_2 \underline{A_2} = \underline{b}$

- $A = \boxed{\text{singular}}$, $\underline{A_1}, \underline{A_2}$ are lin. dependent



$\underline{b} = x_1 A_1$ or $x_2 A_2$

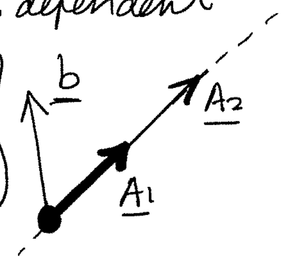
or

$x_1 \underline{A_1} + x_2 \underline{A_2}$

$\boxed{\infty \text{ many sol'ns}}$

or

not on the line made by $\underline{A_1}, \underline{A_2}$



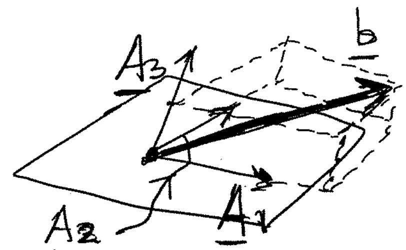
no x_1, x_2 will make

$x_1 \underline{A_1} + x_2 \underline{A_2} = \underline{b}$

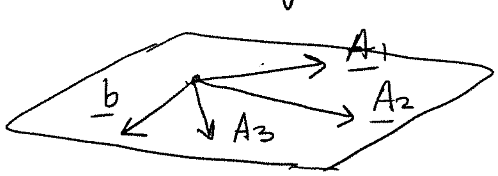
$\boxed{0 \text{ sol'ns}}$

$\underline{R^3}$, i.e. $A = [\underline{A_1}, \underline{A_2}, \underline{A_3}]$

- A = nonsingular
  ($\underline{A_1}, \underline{A_2}, \underline{A_3}$ not in same plane)

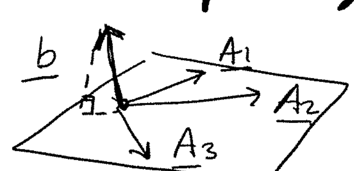


- A = singular (**$\underline{A_1}, \underline{A_2}, \underline{A_3}$ are in the same plane**)



$\underline{A_1}, \underline{A_2}, \underline{A_3}$ & $\underline{b}$ are all in same plane $\Rightarrow$ ∞ many sol'ns

or



$\underline{b}$ is not in the plane with $\underline{A_1}, \underline{A_2}, \underline{A_3}$, $\Rightarrow$ 0 sol'ns.

3 More examples of proofs.

Ex. 8 Recall that in Sec. 1.6 we had a strange property of matrices:

$$(PQ = \mathcal{O}) \underset{\text{in general}}{\not\Rightarrow} (P = \mathcal{O} \text{ or } Q = \mathcal{O})$$

"one cannot cancel by a nonzero matrix".

Now let's prove:

$$(PQ = \mathcal{O} \ \& \ P = \text{nonsingular}) \Rightarrow (Q = \mathcal{O}).$$

Observation: • one can "cancel" by a nonsingular matrix.
• one can cancel by a nonzero number

"a nonsingular matrix is analogous to a nonzero number"
or
"a singular matrix is analogous to a zero number"

Proof: Given (for 2x2 matrices; $P \equiv [\underline{P}_1, \underline{P}_2]$ etc.)

$PQ = \mathcal{O}$

all these mean P = nonsingular:

- $P\underline{x} = \underline{\theta} \Rightarrow \underline{x} = \underline{\theta}$
- $x_1 \underline{P}_1 + x_2 \underline{P}_2 = \underline{\theta} \Rightarrow x_1 = x_2 = 0$
- $P\underline{x} = \underline{b}$ has 1 sol'n

(list all and then pick one that works; may need to use trial-and-error)

Want $Q = \mathcal{O}$.

1) let $Q \equiv [\underline{Q}_1, \underline{Q}_2]$; then

$$PQ = \mathcal{O} \Rightarrow P[\underline{Q}_1, \underline{Q}_2] = \mathcal{O}$$

$$\Rightarrow [P\underline{Q}_1, P\underline{Q}_2] = [\underline{\theta}, \underline{\theta}] \Rightarrow \begin{cases} P\underline{Q}_1 = \underline{\theta} \\ P\underline{Q}_2 = \underline{\theta} \end{cases}$$

2) By "Given, 1st •", $P\underline{Q}_1 = \underline{\theta} \Rightarrow \underline{Q}_1 = \underline{\theta}$.
Similarly, $\underline{Q}_2 = \underline{\theta}$. Then $Q = [\underline{Q}_1, \underline{Q}_2] = [\underline{\theta}, \underline{\theta}] = \mathcal{O}$.

Ex. 9 Let $A, B = n \times n$. Prove that:

(a) $(B = \text{singular}) \Rightarrow (AB = \text{singular})$

Proof: Given (see Def* on p. 6-9)
There is $\underline{x} \neq \underline{\theta}$ s.t. $B\underline{x} = \underline{\theta}$

Want:
There is some $\underline{y} \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta}$.

this special $\underline{x} \neq \underline{\theta}$

Consider $A \cdot (B\underline{x} = \underline{\theta})$

$A(B\underline{x}) = A\underline{\theta} \Rightarrow (AB)\underline{x} = \underline{\theta}$

not $\underline{\theta}$.

Thus, we've found $\underline{y}\ (= \underline{x}) \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta} \Rightarrow$ q.e.d.

(b) **(OPTIONAL)**

$(A = \text{singular}) \Rightarrow (AB = \text{singular})$

Proof: Given
There is $\underline{x} \neq \underline{\theta}$ s.t. $A\underline{x} = \underline{\theta}$

Want:
There is some $\underline{y} \neq \underline{\theta}$ s.t. $(AB)\underline{y} = \underline{\theta}$.

1) Case 1: $B = \text{singular}$.
Then $AB = \text{singular}$ by (a).

2) Case 2: $B = \text{nonsingular}$:
- $B\underline{z} = \underline{\theta} \Rightarrow \underline{z} = \theta$ (Def.)
- $B\underline{z} = \underline{b}$ has 1 sol'n. (Thm. 13)

Take the $\underline{x} \neq \underline{\theta}$ from the "Given": $A\underline{x} = \underline{\theta}$

Consider a l.s. $\boxed{B\underline{z} = \underline{x},}$ where $\underline{x}$ is given, and $\underline{z}$ is to be found. By Thm. 13, such $\underline{z}$ exists (and is unique). Moreover, this $\underline{z} \neq \underline{\theta}$, because $B \cdot \underline{\theta} = \underline{\theta} \neq \underline{x}$.

Now:

$$A \cdot (B\underline{z} = \underline{x})$$

$$(AB)\underline{z} = A\underline{x} \rightarrow \underline{\theta} \text{ (given)}$$

$$(AB)\underline{z} = \underline{\theta}$$

$\neq \underline{\theta}$ (see above).

q.e.d.

Conclusion from Ex. 9(a,b):

$\boxed{(A \text{ or } B = \text{singular}) \Rightarrow (AB = \text{singular})}$

equivalently,

$\boxed{(AB = \text{nonsingular}) \Rightarrow (A \text{ and } B = \text{nonsingular})}$

Compare with the observation on p. 6-15:

$(a \text{ or } b = 0) \Rightarrow (ab = 0) \;\|\; (ab \neq 0) \Rightarrow (a \neq 0, b \neq 0)$

(4) Transpose & (non)singular

Claim: (proof = Extra Credit #3, #57)

$\boxed{(A = \text{nonsingular}) \Leftrightarrow (A^T = \text{nonsingular})}$

Corollary 1

$\boxed{(A = \text{singular}) \Leftrightarrow (A^T = \text{singular})}$

Corollary 2

(**Columns** of A are lin. dep.) $\Leftrightarrow$ (**Rows** of A are lin. dep.)

<u>Proof</u> (OPTIONAL)

$$\begin{pmatrix}\text{Columns of } A\\ \text{are lin. dep.}\end{pmatrix} \underset{\text{Thm. 12}^*}{\Longleftrightarrow} (A = \text{sing.}) \underset{\text{Corollary 1}}{\Longleftrightarrow} (A^T = \text{singular})$$

$$\Longleftrightarrow \begin{pmatrix}\boxed{\text{Columns of } A^T}\\ \text{are lin. dep.}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\boxed{\text{Rows of } A}\\ \text{are lin. dep.}\end{pmatrix}.$$

same

Thus, if A = singular, <u>both</u> its columns <u>and</u> rows are lin. dependent.