Sec. 3.1. Introduction to vector spaces $\mathbb{R}^n$: $\mathbb{R}^2$ and $\mathbb{R}^3$

In this Chapter we'll study so-called vector spaces $\mathbb{R}^n$ and common properties of these spaces. We'll introduce a definition of a vector space later. For now it will suffice to know that $\mathbb{R}^n$ is the set of all vectors will n real components.

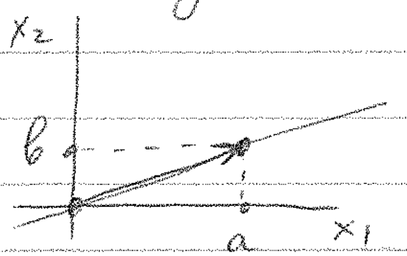
> If you have difficulty imaging a particular situation in $\mathbb{R}^n$ — think how it would work in $\mathbb{R}^2$ and $\mathbb{R}^3$ (the plane and the 3D space).

Most properties of $\mathbb{R}^n$ can be understood by examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.

notation $\rightarrow$ $\mathbb{R}^2 = \{\underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 = \text{real}\}$

Below is a review from Calculus III.

① Eq. of a line in $\mathbb{R}^2$ that goes through $(0,0)$ along vector $\langle a, b \rangle$:



$x_1 = at$
$x_2 = bt$

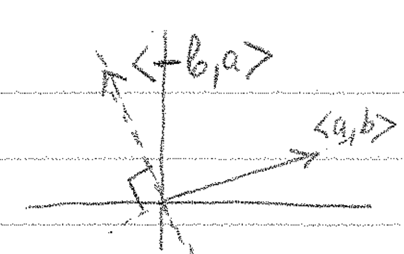
"Usual" notations:
$x = a \cdot t$
$y = b \cdot t$

(Check: $y = bt = b \cdot \frac{x}{a} = \left(\frac{b}{a}\right)x$. ✓)

Notation:

$$\{\underline{x} : \underline{x} = \begin{pmatrix} at \\ bt \end{pmatrix}, \ t = \text{real}\}.$$

② An alternative form of ①.



If $x_1 = at$, $x_2 = bt$, then

$$(-b)\cdot(at) + a\cdot(bt) = 0$$

$$\underbrace{-b}_{A}\cdot x_1 + \underbrace{a}_{B}\cdot x_2 = 0.$$

call this A, B

$$\boxed{A x_1 + B x_2 = 0}$$

$$\boxed{\begin{pmatrix}-b\\ a\end{pmatrix}^T \underline{x} = 0.}$$

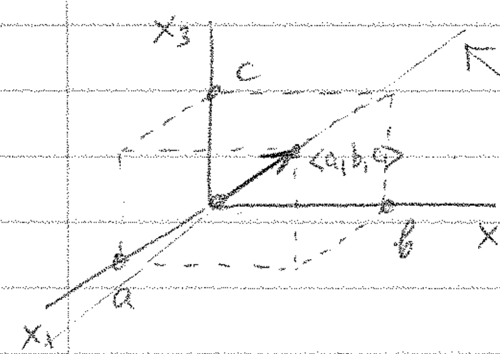
The line is $\perp$ to vector $\begin{pmatrix}-b\\ a\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix}$.

Eq. of a line (in $\mathbb{R}^2$!!!) that goes through (0,0).

③ Eq. of a line in $\mathbb{R}^2$ not through origin:

$$A x_1 + B x_2 + C = 0, \quad C \neq 0.$$

④ Eq. of a line in $\mathbb{R}^3$ going through (0,0,0) and along vector $\langle a, b, c\rangle$:



$$\left\{ \underline{x} : \underline{x} = \begin{pmatrix} at \\ bt \\ ct \end{pmatrix},\ t = \text{real} \right\}.$$

Note: the eq. of a line in 3D CANNOT !!! be written as

$$A x_1 + B x_2 + C x_3 = 0.$$

In fact, we need two equations, not one, to define a line in $\mathbb{R}^3$. See Ex. 2 below.

5 Eq. of a plane (in $\mathbb{R}^3$) that is $\perp$ to vector $\langle A, B, C \rangle$:

$$\boxed{Ax_1 + Bx_2 + Cx_3 = D} \quad (\star)$$

If $D = 0$, the plane goes through the origin.

Ex. 1 What is $\left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_2 = x_1 \right\}$?

Sol'n: Take the constraining equation: $x_2 = x_1$. Put it in form $\star$:

$(-1)\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 = 0$

So this is a plane $\perp \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. $D = 0$, $\Rightarrow$ it goes through the origin.

Geometric interpertation: count "degrees of freedom".

$$\underset{\left(\substack{\text{in } \mathbb{R}^3 \text{ we} \\ \text{have 3 DoF}}\right)}{3} - \underset{\left(\substack{\text{impose 1 eq} \\ = 1 \text{ constraint}}\right)}{1} = \underset{\text{DoFs remain.}}{2}$$

A plane has 2 DoFs, but a line has only 1 DoF.

Ex. 2 What is $\left\{ \underline{x} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, x_1 = x_2 \text{ and } x_2 = -x_3 \right\}$?

Sol'n: Write each constraing eq. in form $\star$:

Ex. 1 $\Rightarrow$ $-1\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 = 0$ ← plane

&

$0\cdot x_2 + 1\cdot x_2 + 1\cdot x_3 = 0$ ← another, non ||, plane.

The "and" says that x belongs to both planes, $\Rightarrow$ it must be ~~along~~ on the intersection line of these two planes.

Thus, the required set is a line in 3D.

Observation: $\boxed{\text{A line in 3D is defined by 2 not 1, equations.}}$

Reason: count DoFs.

$$\underset{\left(\substack{3 \text{ DoFs} \\ \text{in } \mathbb{R}^3}\right)}{3} - \underset{\left(\substack{\text{impose 2} \\ \text{constraining eqs.}}\right)}{2} = \underset{\substack{\text{DoF along} \\ \text{a line}}}{1}$$

HW: 5, 7, 13, 14, 15, 16, 18, 19, 20, 23, 25,
27, 28, 29, 30,
+
2 Word Problems (online).