Ex. 6 Illustration for concept of range.

Find $\mathcal{R}(A)$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Sol'n: $\mathcal{R}(A)$ consists of all possible $A\underline{x}$, where $\underline{x}$ is any vector (in $\mathbb{R}^5$). So:

$$A\underline{x} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \underset{\substack{\uparrow \\ \text{Key Formula}}}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} x_5$$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \qquad 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(x_1 + x_2 + x_4) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(x_3 + 2x_4 + 3x_5)$$

$$\equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} c_2 \quad \text{(with any } c_1, c_2\text{)}.$$

Moral:

- It is easy to see that $A\underline{x} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$, and so $\underline{y} = A\underline{x}$ will have the 3rd component (y_3) equal zero for any $\underline{x}$.
- Therefore, geometrically, $\mathcal{R}(A)$ will be the plane containing all vectors whose 3rd component is zero, i.e., the xy-plane in 3D.