Sec. 4.2. Determinants.

① Determinants & singular matrices.
② Determinants of $n \times n$ matrices
③ Properties of determinants.

① We observed that $(A-\lambda I) = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$ is singular when the quantity

$$(a-\lambda)(d-\lambda) - bc = 0.$$

This quantity is called a <u>determinant</u> of the matrix $(A-\lambda I)$.

E.g., $\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

$$\boxed{\begin{array}{l}\text{So, for } 2\times 2 \text{ matrices, we have:} \\ (A-\lambda I) = \text{singular iff } \det(A-\lambda I) = 0.\end{array}}$$

It turns out that exactly the same statement also holds for $n \times n$ matrix:

$$\boxed{\begin{array}{ll}\text{Thm 3} & \text{Let } A \text{ be } n\times n. \text{ Then} \\ \text{(no proof)} & A = \underline{\text{singular}} \text{ iff } \underline{\det(A) = 0}.\end{array}}$$!

$$\boxed{\begin{array}{l}\text{As applied to the } \underline{\text{eigenvalue}} \text{ problem, we have:} \\ (A-\lambda I) = \text{singular iff } \det(A-\lambda I) = 0.\end{array}}$$

② <u>Determinants of $n\times n$ matrices</u>

We'll give a <u>recursive</u> definition of the determinant of a $(n \times n)$ matrix.

Know $\det(2\times2) \longrightarrow$ Calculate $\det(3\times3) \longrightarrow$
$\longrightarrow$ Calculate $\det(4\times4) \longrightarrow$ etc.

Determinant of a 3×3 matrix

Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \text{then}$$

$$\det A = a_{11} \cdot \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13})$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

M_{11} $\qquad \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = M_{12} \qquad \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = M_{13}$

__Ex. 1__
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 5 & 4 & 6 \end{pmatrix}$$

$$M_{11} = \begin{pmatrix} 3 & 0 \\ 4 & 6 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 0 \\ 5 & 6 \end{pmatrix}, \quad M_{13} = \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix}$$

"minor"

- For a $(n \times n)$ matrix A, M_{ij} is the matrix obtained from A by <u>crossing out</u> its <u>row i & column j</u>.

- A more useful quantity than the minor M_{ij} is a **cofactor**
$$\boxed{A_{ij} = (-1)^{i+j} \det M_{ij}.}$$

Note:
$(-1)^{\text{even}} = +1$
$(-1)^{\text{odd}} = -1$

Then for a 3×3 matrix A,
$$\det A = a_{11} \cdot \underset{(-1)^{1+1} \cdot \det(M_{11})}{A_{11}} + a_{12} \cdot \underset{(-1)^{1+2} \cdot \det(M_{12})}{A_{12}} + a_{13} \cdot \underset{(-1)^{1+3} \det(M_{13})}{A_{13}}$$

Ex. 2 Find det A for $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix}$.

Sol'n: $\det A = 1\cdot(-1)^{1+1}\cdot\begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} + 2\cdot(-1)^{1+2}\cdot\begin{vmatrix} 0 & 0 \\ 5 & 7 \end{vmatrix} + 3\cdot(-1)^{1+3}\cdot\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix}$

shorthand notation for det (M_{12})

$= 1\cdot 1\cdot(28-0) + 2\cdot(-1)\cdot(0-0) + 3\cdot 1\cdot(0-20)$

$= \boxed{-32}$

We've used the formula from the bottom of p. 18-2, where the cofactor expansion was done w.r.t. row 1.

Just out of curiosity, let's do the cofactor expansion w.r.t. row 2:

$\underset{0}{a_{21}}\cdot(-1)^{2+1}\cdot\begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} + \underset{4}{a_{22}}\cdot(-1)^{2+2}\cdot\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} + \underset{0}{a_{23}}\cdot(-1)^{2+3}\cdot\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}$

$= 0 + 4\cdot 1\cdot(7-15) + 0 = \boxed{-32}$. Same answer!

This is **not** a coincidence!

In general, for any $n\times n$ matrix, one has:

$$\det A = a_{k1}A_{k1} + a_{k2}A_{k2} + \dots + a_{kn}A_{kn}$$ ← cofactor expansion w.r.t. row k

$$= a_{1m}A_{1m} + a_{2m}A_{2m} + \dots + a_{nm}A_{nm}$$ ← cofactor expansion w.r.t. **column** m

Practical Q: Which row or column should one choose to compute the determinant?

A: **The one with the greatest number of zeros!** (if any)

③ Some properties of determinants

1) $\det(AB) = \det(A) \cdot \det(B)$,
where both A, B are $n \times n$ matrices.

Note: $\boxed{\det(A+B) \neq \det(A) + \det(B).\ !!}$

2) $\det(A^T) = \det(A)$
3) $\det(I) = 1$
4) Effects of elementary row operations on determinants:
a) Suppose matrix B is obtained from A by interchanging any two rows or any two columns. Then $\det B = -\det A$.

Ex. 3(a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc)$

b) (Corollary of (a))
If any two rows or any two columns of A are the same, then $\det A = 0$.

Ex. 3(b) $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$

c) Let r be a scalar and let B be obtained from A by multiplying any one row or any one column by r. Then $\det B = r \cdot \det A$.

Ex. 3(c)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ r\cdot c & r\cdot d \end{vmatrix} = a\cdot rd - b\cdot rc = r(ad-bc)$$

d) Let B be obtained from A by replacing any of its rows by the sum of this row and another row. Then $\det B = \det A$.
The same holds for columns.

Ex. 3(d)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} (a+b) & b \\ (c+d) & d \end{vmatrix} = (a+b)d - b(c+d) =$$
$$ad + \cancel{bd} - bc - \cancel{bd}$$
$$= ad - bc \quad \checkmark$$

e) Combine (c) & (d):

$$\begin{vmatrix} a+rb & b \\ c+rd & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(and can be straighforwardly extended for $n\times n$)