Sec. 1.1 : Intro to matrices and systems of linear equations.

① Applications

In most real-world situations, problems depend on more than 1 variable. Then we have several coupled equations.
The following example will set the stage for this course. And, we will return to it in Project 1 and Chapter 4.

Ex. 1. In a town of 10,000 people, all people belong to one of two groups:

S — those who subscribe to a local newspaper;
N — those who do not subscribe.

Suppose that initially, there are 5,000 in each group:

$$S_0 = 5{,}000 \; ; \qquad N_0 = 5{,}000.$$

Every year, these population follow the rule:

- 70% of Subscribers keep their subscription (~~and~~ so 30% of them give up subscription);
- 50% of Non-subscribers remain such (and so 50% of Non-subscribers become Subscribers).

Q1: what happens to these populations in 1 year?

$$S_1 = 0.7 S_0 + 0.5 N_0 = 0.7 \cdot 5{,}000 + 0.5 \cdot 5{,}000 = 6{,}000$$

$$N_1 = 0.3 S_0 + 0.5 N_0 = 0.3 \cdot 5{,}000 + 0.5 \cdot 5000 = 4{,}000$$

Here, two unknowns S_1 and N_1 depend on two known variables S_0 and N_0 in a simple manner.

Q2: What happens to these populations in 5 years? In 10 years? In 20 years?

We can iterate the above rule:

$S_2 = 0.7 S_1 + 0.5 N_1$, $N_2 = 0.3 S_1 + 0.5 N_1$, etc.

In this way we find:

$S_5 = 6{,}249$, $N_5 = 3{,}751$

$S_{10} = 6{,}250$, $N_{10} = 3{,}750$

$S_{20} = S_{10}$, $N_{20} = N_{10}$.

Looks like these populations stabilize ...

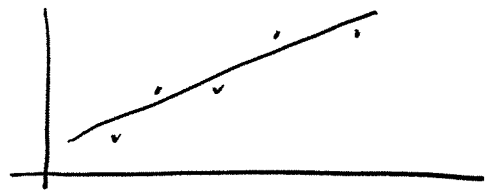
More Qs:

- Is there a more literate way to obtain these answers than the brute-force iterations as above?
- Is there an explanation why these population stabilize at certain values?

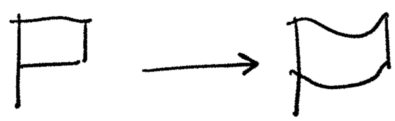
Linear Algebra allows us to answer these Q's.

There are many, many applications of Lin. Algebra.

- Some examples of applications are found in Sec. 1.4 (which we will <u>not</u> cover); e.g., networks (automobile roads, electrical circuits,..)
- Solving differential equations on a computer (<u>project 2</u>)
- Curve fitting: (Sec. 3.8)



- Computer animations: (bonus <u>for Project 4</u>).



<u>DISCLAIMER</u>: Due to time limitations, we won't be able to consider applications of Linear Algebra in class (with very few exceptions). You will see applications in the Projects and your future Engineering & Science Courses.

In this course we will focus on the <u>abstract concepts</u> of vectors, matrices, and the properties that they have.

② <u>Definition and geometric interpretation of linear systems</u> (l.s.)

<u>Def</u>: A <u>l.s.</u> is a <u>set</u> of eqs. where <u>each eq.</u> is <u>linear</u> in each of its unknown variables.

E.g.: $\begin{cases} x_1 + 2x_1x_2 + \sqrt{x_3} = 3 \\ x_1 + 5x_2 - 2x_3 = 6 \end{cases}$ is not a l.s.

because x_1x_2 and $\sqrt{x_3}$ are <u>not</u> linear w.r.t. x_1, x_2, x_3.

However, $\begin{cases} \sqrt{2}\, x_1 - x_2 = \sin 3 \\ 0.1\, x_1 + e^2 \cdot x_2 = \sqrt{5} \end{cases}$ is a l.s.,

because the unknowns x_1, x_2 enter both eqs. linearly.

A general l.s. has the form:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2n}\, x_n = b_2 \\ \vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \dots + a_{mn}\, x_n = b_m. \end{cases}$$

Thus, a_{ij} denotes the coefficient of variable x_j in eq. number i.

The solution of a l.s. is an ordered list $x_1, x_2, \dots x_n$ such that every eq. in the system holds true.

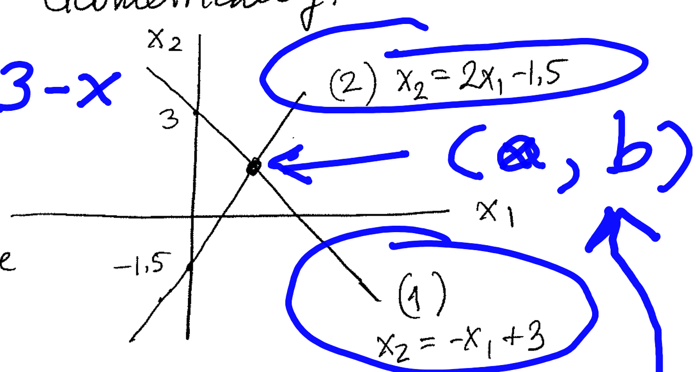
Ex. 2 Consider a l.s.:

$$\begin{cases} x_1 + x_2 = 3 & (1) \\ 2x_1 - x_2 = 1.5 & (2) \end{cases}$$

$\to y = 3 - x$

$x_2 = y$, $x_1 = x$

Geometrically:



Interpretation (important!):

- An eq. with 2 unknowns can be viewed as a straight line in the (x_1, x_2)-plane.
- The solution of a system of 2 such eqs. is the intersection point of two straight lines.

Observation: 2 straight lines in the plane can:

1) intersect at 1 point $\Rightarrow$ 1 solution to l.s.
2) be parallel (no intersection) $\Rightarrow$ 0 soln's to l.s.
3) coincide $\Rightarrow$ ∞ many soln's.

Ex. 3 Consider a l.s. in 3 unknowns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right]$$

(x, y, z)

Each eq. represents a plane in 3D (review Calc. II!).

3 planes can:

1) intersect at 1 point $\Rightarrow$ 1 sol'n to l.s.
2) intersect along a straigh line $\Rightarrow$ ∞ many soln's to l.s.
3) coincide $\Rightarrow$ ∞ many soln's to l.s.
4) any two, or all 3 planes, can be $\parallel$ $\Rightarrow$ 0 soln's to l.s.

From Ex. 2 and 3 there follows the GENERAL FACT: !

- A l.s. can only have 0, 1, or ∞ sol'ns. I.e., it cannot have exactly 2 solutions. If it is known that it has 2 soln's, $\Rightarrow$ it has ∞ many soln's.

3 Matrix representation of a l.s.

Def:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called an $m \times n$ matrix.

Ex. 4

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & -6 \\ 0 & 7 \end{pmatrix}$$

is a 4×2 matrix;

its entry in 4th row & 1st column is: $A_{41} = 0$.

Ex.5 Consider a 3x3 l.s.:
$$\begin{aligned} 2x_1+x_2-x_3 &= 6 \\ x_1-x_2+4x_3 &= 1 \\ 5x_2-x_3 &= 7 \end{aligned}$$

The coefficient matrix of this l.s. is: $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 4 \\ 0 & 5 & -1 \end{pmatrix}$

To account for the constants on the right-hand side (r.h.s.), we define the augmented matrix of the l.s.:

$$B = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & -1 & 4 & 1 \\ 0 & 5 & -1 & 7 \end{array}\right)$$

For a general l.s.
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{mn}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$
the augmented matrix is:

$$B = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \equiv [A \mid \underline{b}]$$

the coefficient matrix $\rightarrow A$ $\qquad$ $\underline{b} \leftarrow$ the column of r.h.s.

④ Elementary operations with l.s. & matrices

Def: Two l.s. are called equivalent if they have the same solution.

Ex.6 l.s.1: $\begin{aligned} x_1+x_2+x_3 &= 3 \\ x_1+x_2 &= 2 \end{aligned}$ | l.s.2: $\begin{aligned} 2x_1+2x_2+x_3 &= 5 \\ x_3 &= 1 \end{aligned}$

These l.s. are equivalent since both have the solution:

$$\begin{aligned} x_1 &= 2-x_2 \\ x_3 &= 1. \end{aligned}$$

In fact, this can be viewed as the simplest (of the three) l.s. that is equivalent to l.s. 1&2.

We now consider ways of reducing a given l.s. to the simplest possible form. During such a reduction process, we must obtain l.s. that is equivalent to the l.s. at the previous step (and hence to the original l.s.).

Thm. 1 The following elementary operations (EO) on equations (i.e. rows) of a l.s. result in an equivalent l.s.:
1) Interchanging any two eqs.
2) Multiplying any equation by a nonzero number.
3) Adding (a constant multiple of) one eq. to another.

$$x_1+x_2=2$$
$$3x_1+3x_2=6$$

In Ex. 5 we observed that a l.s. is completely described by its augmented matrix. Therefore, instead of performing the above EO on eqs., we'll perform them on the rows of the augmented matrix.

Ex. 7 Solve the l.s.

$$\begin{aligned} x_1 + x_2 \qquad &= 1 \\ 2x_1 + x_2 + x_3 &= 2 \\ -x_1 \qquad + 2x_3 &= -4. \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 0 & 2 & -4 \end{array}\right)$$

We'll first show how to transform this to the simplest form using equations. This is done only to illustrate how this process will lead to the solution of the l.s. Then we'll repeat the same process using the augmented matrix of this l.s., and from that point on you'll have to use only the augmented matrix to solve any l.s. in this course!

Ex. 7 Gauss-Jordan elimination

(see also Ex. 7 in textbook)

L.S.	Augmented matrix	Comments
$x_1 + x_2 = 1$ $2x_1 + x_2 + x_3 = 2$ $-x_1 + 2x_3 = -4$	$\left(\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 0 & 2 & -4 \end{array}\right)$	Use Eq. 1 to eliminate x_1 from Eqs. 2 and 3: $E_2 - 2E_1 \to E_2$ $E_3 + E_1 \to E_3$ $2 - 2 \cdot 1 = 0$
$x_1 + x_2 = 1$ $-x_2 + x_3 = 0$ $x_2 + 2x_3 = -3$	$\left(\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -3 \end{array}\right)$	1) From this moment, do NOT use Eq. 1 until further notice 2) Make the coefficient of x_2 in Eq. 2 a "1". $\begin{array}{ccc\|c} 2 & 1 & 1 & 2 \\ -2 & 2 & 0 & 2 \\ \hline 0 & -1 & 1 & 0 \end{array}$
$x_1 + x_2 = 1$ $x_2 + 2x_3 = -3$ $-x_2 + x_3 = 0$	$\left(\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & -1 & 1 & 0 \end{array}\right)$	Use Eq. 2 to eliminate x_2 from Eq. 3 (= "all eqs. below Eq. 2"). $E_3 + E_2 \to E_3$ $\begin{array}{cc\|c} 1 & 2 & -3 \\ -1 & 1 & 0 \\ \hline 0 & 3 & -3 \end{array}$
$x_1 + x_2 = 1$ $x_2 + 2x_3 = -3$ $3x_3 = -3$	$\left(\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & -3 \end{array}\right)$	1) Do NOT ~~not~~ use Eq. 2 until further notice (which will come soon) 2) Make the coefficient of x_3 in Eq. 3 a "1" $E_3/3 \to E_3$
$x_1 + x_2 = 1$ $x_2 + 2x_3 = -3$ $x_3 = -1$ You've found x_3.	$\left(\begin{array}{ccc\|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$	Use Eq. 3 to eliminate x_3 from all eqs. above Eq. 3 (i.e., from Eqs. 1 and 2). $E_2 - 2E_3 \to E_2$ $E_1 \to E_1$

$$\begin{array}{ccc|c} 0 & 1 & 2 & -3 \\ -\;0 & 0 & 2 & -2 \\ \hline 0 & 1 & 0 & -1 \end{array}$$

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_2 \quad &= -1 \\ x_3 &= -1 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

You've now found x_2.

1) From now on, never use Eq. 3 again.

2) Use Eq. 2 to eliminate x_2 from all eqs. above it (which in this case is just Eq. 1).

$E_1 - E_2 \rightarrow E_1$

$$\Downarrow$$

$$\begin{aligned} x_1 \quad &= 2 \\ x_2 \quad &= -1 \\ x_3 &= -1 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

You've now found all your variables!

DONE.

HW: 1, 5, 6, 9, 11, 15, 21, 29 (up to Ex. 3 on p. 1-5) 31, 33, 35