Sec. 1.5 Matrix operations

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(1) Elementary operations

Def: let A be $m \times n$ matrix; and let B be $r \times s$ matrix. They are said to be equal if:

- their dimensions are the same ($m=r$, $n=s$);
- their corresponding entries are equal.

Ex. 1 (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ (second condition is violated: order of entries matters!)

(b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix}$ (first condition is violated: the dimensions do not match)

One adds matrices similarly how one adds scalars.

Def: If A and B are matrices of the same dimensions, then one finds $(A+B)$ simply by adding their corresponding entries.

- Note: One cannot add matrices if their dimensions are not equal.

Ex. 2 (a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \end{pmatrix}$,

$C = \begin{pmatrix} 21 & 22 \\ 23 & 24 \\ 25 & 26 \end{pmatrix}$.

$$A+B = \begin{pmatrix} 1+11 & 2+12 & 3+13 \\ 4+14 & 5+15 & 6+16 \end{pmatrix},$$

but cannot compute $A+C$ and $B+C$.

(b) Find D s.t. $A+D=B$.

Solution: $D = B-A = \begin{pmatrix} 11-1 & 12-2 & 13-3 \\ 14-4 & 15-5 & 16-6 \end{pmatrix}$.

Def: To multiply matrix A by scalar r, simply multiply each entry of A by r.

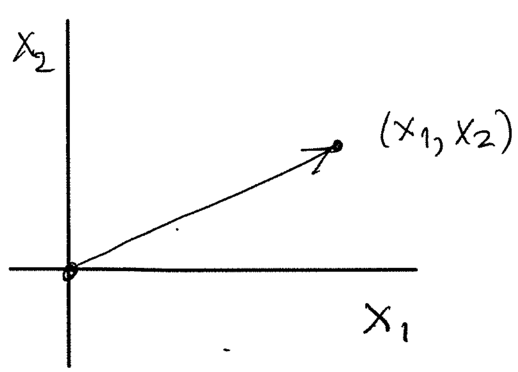
Ex. 3

$$11 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 11 & 22 \\ 33 & 44 \\ 55 & 66 \end{pmatrix}.$$

Equipped with these definitions, we can construct any linear combination of matrices:

1/25/19 $r \cdot A + s \cdot B$.

(2) Vectors in $\mathbb{R}^n$



Recall from Calculus that by convention, the starting point of any vector by default is at the origin.

Then a vector is fully defined by specifying its end point coordinates, x_1 & x_2.

So: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

In the same way, one defines <u>any ordered list</u> of numbers to be a vector:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leftarrow \text{"n-dimensional vector"}$$

A collection of all such vectors is $\mathbb{R}^n$.

"real"

Formal writing:

$$\mathbb{R}^n = \left\{ \underline{x} : \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ x_1, x_2, \dots x_n \text{ are all real numbers} \right\}.$$

MUST READ ON YOUR OWN

Ex. 2, 3 in textbook about the <u>vector form</u> of the solution of a l.s.

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3 Matrix-vector multiplication

It is defined to provide a convenient tool for writing down a l.s. in compact form.

Want to mimic a single equation:

$$a \cdot x = b$$

Coefficient unknown constant

Now consider a l.s.

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

It has:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

coefficient matrix — unknown vector — const. vector

So, mimicking the single-eq. case, we want to write the l.s. as:

$$A \cdot \underline{x} = \underline{b}$$

This will be so if we define matrix-vector multiplication as:

$$\begin{pmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} \cdot x_1 + \boxed{a_{12}} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{pmatrix}$$

In "Sigma-notation":

$$a_{11} x_1 + a_{12} x_2 = \sum_{j=1}^{2} a_{1j} x_j,$$

or more generally:

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = \sum_{j=1}^{n} a_{1j} x_j .$$

So:

$$A \cdot \underline{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}.$$

Then the general l.s.

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \end{cases}$$

can be written as

$$\boxed{A \cdot \underline{x} = \underline{b},}$$ as desired.

See Ex. 6 in book for numbers.

④ Matrix-matrix multiplication

Let A be $m \times n$, B be $n \times s$.

n and n: same

Note that:

$$B \equiv \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} = [\underline{b}_1, \underline{b}_2, \ldots \underline{b}_s],$$

where

$$\underline{b}_k = \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{pmatrix} \leftarrow k\text{th column of } B.$$

Then $$\boxed{AB \equiv A[\underline{b}_1, \underline{b}_2, \ldots \underline{b}_s] = [A\underline{b}_1, A\underline{b}_2, \ldots A\underline{b}_s]}$$

i.e., we simply multiply each column of B, $\underline{b}_k$, by A.

Thus, using our knowledge of matrix-vector multiplication, we can present matrix-matrix multiplication as:

$$(A \cdot B)_{\ell k} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{\ell 1} & \cdots & a_{\ell n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1k} & \cdots & b_{1s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & & b_{nk} & & b_{ns} \end{pmatrix} =$$

(ℓ, k)-th entry

$$= \sum_{j=1}^{n} a_{\ell j} b_{jk}$$

$a_{\ell 1} b_{1k} + a_{\ell 2} b_{2k} + \ldots a_{\ell s} b_{sk}$

<u>Mnemonically</u>: "multiply l-th <u>row</u> of the 1st matrix by the k-th <u>column</u> of the 2nd".

<u>Ex. 4</u> Given matrices

$$A=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix},\quad B=\begin{pmatrix}-3 & 2\\ 1 & -2\end{pmatrix},$$

$$C=\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 1\end{pmatrix},\quad D=\begin{pmatrix}3 & 1\\ -1 & -2\\ 1 & 1\end{pmatrix},$$

compute:

AB, BA, AC, CA, CD, DC.

<u>Sol'n:</u> (a) $\underset{2\times2}{A}\,\underset{2\times2}{B}=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}-3 & 2\\ 1 & -2\end{pmatrix}$

$$=\begin{pmatrix}1\cdot(-3)+2\cdot1 & 1\cdot2+2\cdot(-2)\\ 4\cdot(-3)+3\cdot1 & 4\cdot2+3\cdot(-2)\end{pmatrix}=\begin{pmatrix}-1 & -2\\ -9 & 2\end{pmatrix}$$

(b) $BA=\begin{pmatrix}-3 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}=$

$$=\begin{pmatrix}-3\cdot1+2\cdot4 & -3\cdot2+2\cdot3\\ 1\cdot1+(-2)\cdot4 & 1\cdot2+(-2)\cdot3\end{pmatrix}=\begin{pmatrix}5 & 0\\ -7 & -4\end{pmatrix}$$

• (c) $AC=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 0\\ 4 & 3 & -5\end{pmatrix}$ 2×3

$2\times\boxed{2}$ $\boxed{2}\times3$ same

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• (d) $CA=\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\ldots$ CANNOT MULTIPLY!

$2\times\boxed{3} \neq \boxed{2}\times2$ dimensions do not match

(e) $CD = \begin{pmatrix} \boxed{1 \quad 0 \quad -2} \\ 0 \quad 1 \quad 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{pmatrix} =$

$2 \times 3 \quad \checkmark \text{ same} \quad 3 \times 2 \quad \rightarrow \quad 2 \times 2$

$$= \left(\begin{array}{c|c} 1\cdot 3 + 0\cdot(-1) + (-2)\cdot 1 & 1\cdot 1 + 0\cdot(-2) + (-2)\cdot 1 \\ \hline 0\cdot 3 + 1\cdot(-1) + 1\cdot 1 & 0\cdot 1 + 1\cdot(-2) + 1\cdot 1 \end{array} \right) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

(f) $DC = \begin{pmatrix} 3 & 1 \\ -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} =$

$3 \times 2 \quad \checkmark \text{ same} \quad 2 \times 3 \quad \rightarrow \quad 3 \times 3$

$$\left(\begin{array}{c|c|c} 3\cdot 1 + 1\cdot 0 & 3\cdot 0 + 1\cdot 1 & 3\cdot(-2) + 1\cdot 1 \\ \hline -1\cdot 1 + (-2)\cdot 0 & -1\cdot 0 + (-2)\cdot 1 & (-1)(-2) + (-2)\cdot 1 \\ \hline 1\cdot 1 + 1\cdot 0 & 1\cdot 0 + 1\cdot 1 & 1\cdot(-2) + 1\cdot 1 \end{array} \right) = \begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

<u>Observation:</u> In general, for matrices:

$$AB \neq BA.$$

- AB may exist (c), but BA does not (d).
- Both AB and BA exist, but have different dimensions (e, f).
- AB and BA exist and have the same dimensions, but their entries are different (a, b).

MUST READ ON YOUR OWN:
Ex. 5 in textbook + half a page right after it (about: expressing a l.s. in matrix form.)

⑤ Alternative formulation of matrix multiplication

Let's look at a l.s. in matrix form (see topic ③ and the must-read Ex. 5 in book):

Ex. 5

$$\underset{A}{\underbrace{\begin{pmatrix} \boxed{\begin{matrix}1\\4\end{matrix}} & \boxed{\begin{matrix}2\\5\end{matrix}} & \boxed{\begin{matrix}3\\6\end{matrix}} \end{pmatrix}}_{2\times 3}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{3\times 1,\ \underline{x}} = \underbrace{\begin{pmatrix} 7 \\ 8 \end{pmatrix}}_{\underline{b}}$$

$\underbrace{\underline{A_1} \quad \underline{A_2} \quad \underline{A_3}}$ ← columns of A

linear combination of $\underline{A_1}, \underline{A_2}, \underline{A_3}$

$$1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3 = 7$$
$$4\cdot x_1 + 5\cdot x_2 + 6\cdot x_3 = 8, \quad \text{or}$$

$$\boxed{\begin{pmatrix} 1 \\ 4 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \cdot x_2 + \begin{pmatrix} 3 \\ 6 \end{pmatrix} x_3 = \begin{pmatrix} 7 \\ 8 \end{pmatrix}}$$

From this we derive 2 important conclusions:

$\boxed{1}$

$$A \cdot \underline{x} \equiv [\underline{A_1}, \underline{A_2}, \underline{A_3}] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= \underline{A_1} \cdot x_1 + \underline{A_2} \cdot x_2 + \underline{A_3}\, x_3$$

In general, if A is $m \times n$ and $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then

$$\boxed{A \cdot \underline{x} = x_1 \underline{A_1} + x_2 \underline{A_2} + \dots + x_n \underline{A_n}} \ !$$

Key Formula

MUST MEMORIZE!

See Ex. 7 in textbook, Thm. 5, and the example after it for further illustration of the Key Formula.

$\boxed{2}$ (Corollary of the Key Formula)

If $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a solution of a l.s. $A \cdot \underline{x} = \underline{b}$, then:

$$x_1 \underline{A_1} + x_2 \cdot \underline{A_2} + \dots + x_n \underline{A_n} = \underline{b}.$$

In words: $\underline{b}$ is a linear combination of the columns of A.

For example, in the above Ex. 5, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with $x_1 = -13/3$, $x_2 = 8/3$, $x_3 = 2$ is a solution of the l.s. (one of ∞ many, as per Corollary of Thm. 3 (p. 3-2 of Notes for Sec. 1.3)), then

$$\begin{array}{ccccccccccc} -\frac{13}{3} & \cdot & \begin{pmatrix} 1 \\ 4 \end{pmatrix} & + & \frac{8}{3} \cdot & \begin{pmatrix} 2 \\ 5 \end{pmatrix} & + & 2 \cdot & \begin{pmatrix} 3 \\ 6 \end{pmatrix} & = & \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ x_1 & \cdot & \underline{A_1} & + & x_2 \cdot & \underline{A_2} & + & x_3 \cdot & \underline{A_3} & = & \underline{b} \end{array}$$

⑥ Solving a "matrix equation"

Ex. 6 Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4-10

Find B s.t. $AB = C$.

Sol'n: 1) Analyze dimensions.

$$\underset{2\times 2}{A} \cdot \underset{m\times n}{B} = \underset{2\times 2}{C} \Rightarrow n = 2$$

$\hookrightarrow m = 2$

Thus B must be 2×2, and so we can write $B = [\underline{B}_1, \underline{B}_2]$ 2×1 columns of B.

2) Similarly, $C = [\underline{C}_1, \underline{C}_2] \equiv \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$

Then $AB = C$ can be written as

$$A[\underline{B}_1, \underline{B}_2] = [\underline{C}_1, \underline{C}_2] \Rightarrow [A\underline{B}_1, A\underline{B}_2] = [\underline{C}_1, \underline{C}_2]$$

$$\Rightarrow A\underline{B}_1 = \underline{C}_1 \text{ and } A\underline{B}_2 = \underline{C}_2.$$

Thus, we need to solve two l.s.

3) $A\underline{B}_1 = \underline{C}_1$

$\hookrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$$\left(\begin{array}{cc|c}1 & 1 & 1\\1 & 2 & 0\end{array}\right) \xrightarrow{REF} \left(\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & -1\end{array}\right) \Rightarrow \underline{B}_1 = \begin{pmatrix}2\\-1\end{pmatrix}$$

4) $A\underline{B}_2 = \underline{C}_2$:

$$\left(\begin{array}{cc|c}1 & 1 & 0\\1 & 2 & 1\end{array}\right) \xrightarrow{REF} \left(\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\end{array}\right) \Rightarrow \underline{B}_2 = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\Rightarrow B = [\underline{B}_1, \underline{B}_2] = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}.$$

1/30/19 See also Thm. 6 in book about the same method.