University of Vermont
EE 215: Electric Energy System Analysis
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Notes on Newton-Raphson Power Flows
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Given a power system with n buses, let $E_i := V_i\angle\theta_i$ be the phasor voltage at bus i, and entry (i,k) in the admittance matrix is defined as: $[\boldsymbol{Y}_{bus}]_{ik} = Y_{ik} := G_{ik} + \jmath B_{ik}$, where $Y_{ik} \equiv 0$ when nodes i and k have no direct connection. Assume at each node a known complex power $S_i^{inj} = P_i^{inj} + \jmath Q_i^{inj}$ is injected into bus i. Then, assume the slack node is labeled as bus 1, so that we can define vectors:

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} V_2 \\ \vdots \\ V_n \end{bmatrix} \Rightarrow \boldsymbol{x} = \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{V} \end{bmatrix} \tag{1}$$

Then, with $\theta_{ik} := \theta_i - \theta_k$, the power flow equations yield:

$$P_i^{inj} = V_i \sum_{k=1}^{n} V_k (G_{ik}\cos(\theta_{ik}) + B_{ik}\sin(\theta_{ik})) := P_i(\boldsymbol{x}) \qquad i = 1, 2, \dots, n \tag{2}$$

$$Q_i^{inj} = V_i \sum_{k=1}^{n} V_k (G_{ik}\sin(\theta_{ik}) - B_{ik}\cos(\theta_{ik})) := Q_i(\boldsymbol{x}) \qquad i = 1, 2, \dots, n \tag{3}$$

$$\tag{4}$$

where $P_i(\boldsymbol{x}), Q_i(\boldsymbol{x})$ are functions of unknown $\boldsymbol{x}$. The goal of N-R algorithm is to iteratively pick a sequence of $\boldsymbol{x}^v$ to drive the mismatches to zero to get a solution $\boldsymbol{x}$ such that:

$$P_i^{inj} = P_i(\boldsymbol{x}) \qquad i = 2, \dots, n \tag{5}$$

$$Q_i^{inj} = Q_i(\boldsymbol{x}) \qquad i = 2, \dots, n \tag{6}$$

$$\tag{7}$$

where we have removed the first active and reactive equations associated with the slack bus.

To set up the N-R $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, we subtract knowns and unknowns and get:

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} P_2(\boldsymbol{x}) - P_2^{inj} \\ \vdots \\ P_n(\boldsymbol{x}) - P_n^{inj} \\ Q_2(\boldsymbol{x}) - Q_2^{inj} \\ \vdots \\ Q_n(\boldsymbol{x}) - Q_n^{inj} \end{bmatrix} = \begin{bmatrix} -\Delta P_2(\boldsymbol{x}) \\ \vdots \\ -\Delta P_n(\boldsymbol{x}) \\ -\Delta Q_2(\boldsymbol{x}) \\ \vdots \\ -\Delta Q_n(\boldsymbol{x}) \end{bmatrix} = \boldsymbol{0}. \tag{8}$$

where $\Delta \boldsymbol{P}(\boldsymbol{x}) := \boldsymbol{P}^{inj} - \boldsymbol{P}(\boldsymbol{x})$ and $\Delta \boldsymbol{Q}(\boldsymbol{x}) := \boldsymbol{Q}^{inj} - \boldsymbol{Q}(\boldsymbol{x})$ are the mismatch vectors.

Now, the Jacobian $\boldsymbol{J}$ is defined by the four $(n-1) \times (n-1)$ sub-matrices of partial derivatives as:

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{V}} \\ \frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{V}} \end{bmatrix} \tag{9}$$

where off-diagonal entries of the sub matrices are defined for $i \neq k$ as:

$$\left[\frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{\theta}}\right]_{ik} = \frac{\partial P_i(\boldsymbol{x})}{\partial \theta_k} = V_i V_k \left(G_{ik}\sin(\theta_{ik}) - B_{ik}\cos(\theta_{ik})\right) \tag{10}$$

$$\left[\frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{V}}\right]_{ik} = \frac{\partial P_i(\boldsymbol{x})}{\partial V_k} = V_i \left(G_{ik}\cos(\theta_{ik}) + B_{ik}\sin(\theta_{ik})\right) \tag{11}$$

$$\left[\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{\theta}}\right]_{ik} = \frac{\partial Q_i(\boldsymbol{x})}{\partial \theta_k} = -V_i V_k \left(G_{ik}\cos(\theta_{ik}) + B_{ik}\sin(\theta_{ik})\right) \tag{12}$$

$$\left[\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{V}}\right]_{ik} = \frac{\partial Q_i(\boldsymbol{x})}{\partial V_k} = V_i \left(G_{ik}\sin(\theta_{ik}) - B_{ik}\cos(\theta_{ik})\right) \tag{13}$$

and the diagonal entries $i = k$ as follows:

$$\left[\frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{\theta}}\right]_{ii} = \frac{\partial P_i(\boldsymbol{x})}{\partial \theta_i} = -Q_i(\boldsymbol{x}) - B_{ii}V_i^2 \tag{14}$$

$$\left[\frac{\partial \boldsymbol{P}(\boldsymbol{x})}{\partial \boldsymbol{V}}\right]_{ii} = \frac{\partial P_i(\boldsymbol{x})}{\partial V_i} = \frac{P_i(\boldsymbol{x})}{V_i} + G_{ii}V_i \tag{15}$$

$$\left[\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{\theta}}\right]_{ii} = \frac{\partial Q_i(\boldsymbol{x})}{\partial \theta_i} = P_i(\boldsymbol{x}) - G_{ii}V_i^2 \tag{16}$$

$$\left[\frac{\partial \boldsymbol{Q}(\boldsymbol{x})}{\partial \boldsymbol{V}}\right]_{ii} = \frac{\partial Q_i(\boldsymbol{x})}{\partial V_i} = \frac{Q_i(\boldsymbol{x})}{V_i} - B_{ii}V_i \tag{17}$$

Then, using the Jacobian, known PQ injections, PV bus voltage set-points, and slack bus, we want to solve for $\Delta \boldsymbol{x}^v := \boldsymbol{x}^{v+1} - \boldsymbol{x}^v$ by using the N-R method:

$$\boldsymbol{J}(\boldsymbol{x}^v)\Delta \boldsymbol{x}^v = -\boldsymbol{f}(\boldsymbol{x}^v) \tag{18}$$

within the context of the power flow gives:

$$\begin{bmatrix} \frac{\partial \boldsymbol{P}(\boldsymbol{x}^v)}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{P}(\boldsymbol{x}^v)}{\partial \boldsymbol{V}} \\ \frac{\partial \boldsymbol{Q}(\boldsymbol{x}^v)}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{Q}(\boldsymbol{x}^v)}{\partial \boldsymbol{V}} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{\theta}^v \\ \Delta \boldsymbol{V}^v \end{bmatrix} = \begin{bmatrix} \Delta \boldsymbol{P}(\boldsymbol{x}^v) \\ \Delta \boldsymbol{Q}(\boldsymbol{x}^v) \end{bmatrix}. \tag{19}$$

A couple comments for Equation (19)

- Given an iterate or initial guess $\boldsymbol{x}^v$, we can calculate corresponding mismatches on the RHS and as we iterate towards solutions, these mismatches should go to zero (i.e. converge).
- The iterates are updated with $\boldsymbol{x}^{v+1} = \Delta \boldsymbol{x}^v + \boldsymbol{x}^v$ at which point you can update the mismatches and the Jacobian and continue iterating (i.e. $v + 1 \to v$).
- The generators at PV buses specify V_i but not Q_i^{inj} (note: any loads at PV buses can still inject/consume complex power). This means we can reduce dimensionality of $\boldsymbol{J}$. For example, assume bus 2 is a PV bus and slack bus is bus 1. Then we know V_1, V_2, so there is no need to include V_2 in N-R iterative scheme, because after solving for $\boldsymbol{x}$, we would know $\boldsymbol{\theta}$ and $V_3, \ldots, V_n$, so we can compute $Q_1(\boldsymbol{x}), Q_2(\boldsymbol{x})$. The changes to the N-R method is therefore
  - Remove V_2 from $\boldsymbol{V}$
  - Remove $\Delta Q_2(\boldsymbol{x})$ from RHS of (19)

– Remove corresponding row (i.e. $[\frac{\partial Q_2}{\partial \boldsymbol{\theta}} \frac{\partial Q_2}{\partial \boldsymbol{V}}]$) and column (i.e. $[\frac{\partial \boldsymbol{P}}{\partial V_2}; \frac{\partial \boldsymbol{Q}}{\partial V_2}]$) of the full $\boldsymbol{J}$

- After converging to a solution, it is important to check reactive injections required from generators at each PV bus (i.e. need $Q_i(\boldsymbol{x}) - Q_i^{inj} < Q_i^{\max}$). If reactive injections from generators exceed $Q_i^{\max}$, then bus i's type switches from PV to PQ with reactive injections from generators equal to $Q_i^{\max}$ and you continue iterating with voltages (e.g. V_2) as unknown variables in N-R. The case for checking the reactive lower limit $Q_i^{\min}$ is similar.

- The change of a bus from PV to PQ label is permanent for the remainder of the iterative scheme and the bus-label switch should be output by a Power Flow Solver during the iterative process as it is important to know which buses (or generators) are unable to maintain the desired voltage magnitude.