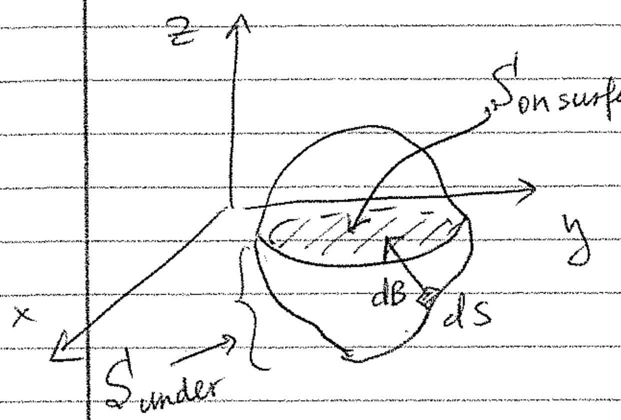
Application of Divergence Thm:
Archimedes Law of Buoyancy.



Will show:

$$\vec{B} = \rho g V_{\text{submerged}}$$

buoyancy force; ρ density of fluid

(= weight of fluid in $V_{\text{subm.}}$)

1) Force on a small surface element:

$$d\vec{B} = p \cdot (-\vec{n}) \cdot dS$$

p pressure; $\vec{n}$ unit normal vector pointing <u>inside</u>

2) $$\vec{B} = \iint_{S_{\text{under}}} d\vec{B} = \iint_{S_{\text{under}}} p \cdot (-\vec{n})\, dS = \iint_{S_{\text{under}}} \underbrace{\rho g (-z)}_{p} (-\vec{n})\, dS$$

$\vec{n} = \langle n_x, n_y, n_z \rangle$

$$= \iint_{S_{\text{under}}} \rho g z \langle n_x, n_y, n_z \rangle\, dS \equiv \langle B_x, B_y, B_z \rangle.$$

3) $n_x = \vec{i} \cdot \vec{n}$ $\left(= \langle 1,0,0 \rangle \cdot \langle n_x, n_y, n_z \rangle \right)$.

Sec. 12.3

$$n_y = \vec{j} \cdot \vec{n}, \quad n_z = \vec{k} \cdot \vec{n}.$$

Then $$B_x = \rho g \left(\iint_{S_{\text{under}}} z\, n_x\, dS + \iint_{S_{\text{on surface}}} \underbrace{z}_{=0 \text{ in } xy\text{-plane}} \cdot n_x \cdot dS \right) =$$

$$= \rho g \oiint_{S_{submerged}} z\, n_x\, dS = \rho g \oiint_{S_{subm.}} \underbrace{z\,\vec{i}}_{\text{some } \vec{F_x}.} \bullet \vec{n}\, dS$$

This is now in the form to which we can apply the Div. Thm:

$$\oiint_{S_{subm}} \vec{F_x} \bullet \vec{n}\, dS = \iiint_{V_{subm.}} \operatorname{div} \vec{F_x}\, dV$$

$$\operatorname{div} \vec{F_x} = \operatorname{div} \langle z, 0, 0 \rangle = \frac{\partial z}{\partial x} + 0 + 0 = 0.$$

So $\boxed{B_x = \rho g \oiint_{subm} z \cdot \vec{i} \bullet \vec{n}\, dS = \rho g \iiint_{V_{subm}} \operatorname{div}(z\vec{i})\, dV = 0.}$

Similarly, $\boxed{B_y = 0}$.

Finally, $B_z = \rho g \oiint_{S_{subm.}} z \cdot \vec{k} \bullet \vec{n}\, dS$

$$= \rho g \cdot \iiint_{V_{subm.}} \underbrace{\operatorname{div} \langle 0, 0, z \rangle}_{0 + 0 + \frac{\partial z}{\partial z} = 1} dV = \rho g \iiint_{V_{subm.}} 1 \cdot dV = \rho g V_{subm.}$$

Thus,
$$\vec{B} = \langle 0, 0, \rho g V_{subm.} \rangle,$$

q.e.d.