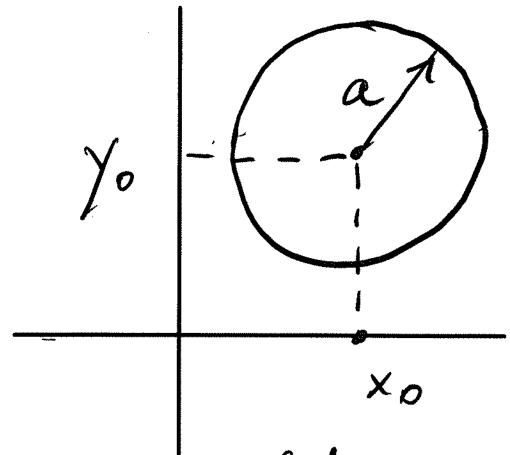
Sec. 12.6 Review of ellipses and hyperbolas; Cylinders

① Eqs. of an ellipse (in 2D)



Circle:

radius $= a$

center @ (x_0, y_0)

Equation:

$$(x-x_0)^2 + (y-y_0)^2 = a^2$$

In the remainder of this section, let $(x_0, y_0) = (0,0)$. Then

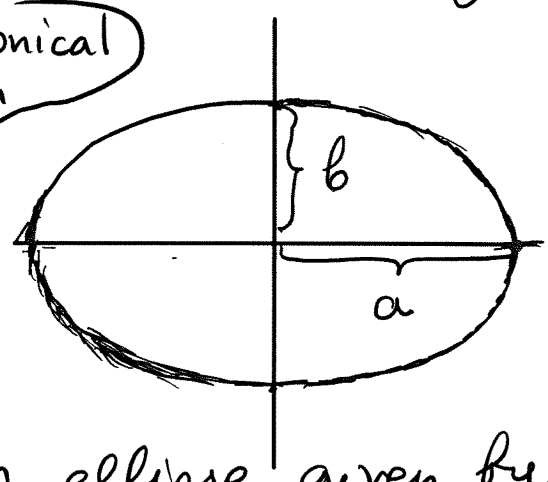
$$x^2 + y^2 = a^2 \iff$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

The equation of an ellipse is more general:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad (*) \leftarrow \text{Canonical form}$$

a, b are semi-axes of the ellipse.



Question:

How can we draw an ellipse given by

$$Ax^2 + By^2 = C \quad (A, B, C > 0) \ ?$$

Answer: We must first put it in the canonical form and from it, find a & b.

- Make a "1" on the right-hand side:

$$\frac{Ax^2}{C} + \frac{By^2}{C} = 1$$

- Move A, B in the denominator:

$$\frac{x^2}{(C/A)} + \frac{y^2}{(C/B)} = 1.$$

Therefore, $a^2 = C/A$, $b^2 = C/B$, $\Rightarrow$

$$a = \sqrt{C/A}, \quad b = \sqrt{C/b}.$$

Parametric eqs. of an ellipse given by (*):

$$\boxed{x = a\cdot\cos t, \quad y = b\cdot\sin t, \quad 0 \le t \le 2\pi}$$

MUST MEMORIZE and USE ONLY THEM for PLOTTING ELLIPSES!

Why are these eqs. correct? Sub into (*):

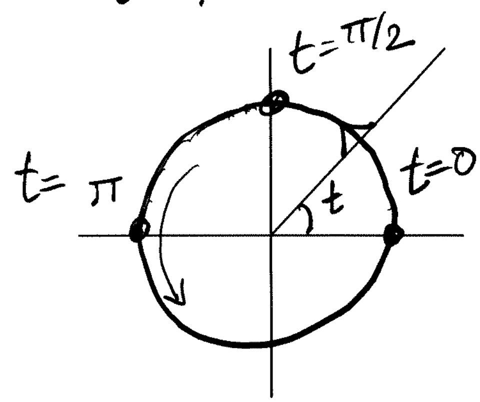
$$\frac{(a\cos t)^2}{a^2} + \left(\frac{b\sin t}{b}\right)^2 = \frac{\cancel{a^2}\cos^2 t}{\cancel{a^2}} + \frac{\cancel{b^2}\operatorname{sen}^2 t}{\cancel{b^2}} =$$

$$\cos^2 t + \sin^2 t = 1. \quad \checkmark.$$

The above range of t:

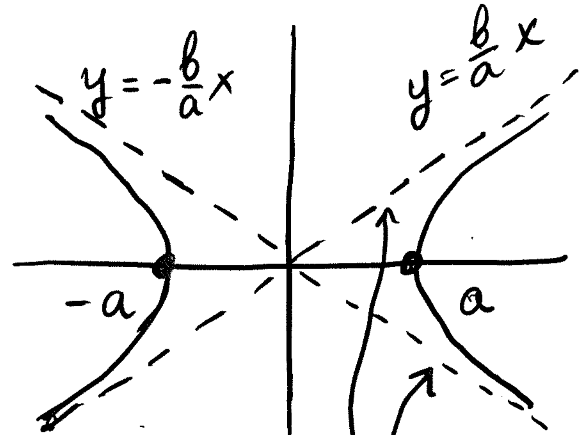
$0 \le t \le 2\pi$

corresponds to:



2 Hyperbola in 2D

Canonical equations:



(**-I) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

For x, y "very large", the "1" on the r.h.s. almost doesn't matter,

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} \approx 0 \Rightarrow y^2 \approx \frac{b^2}{a^2}x^2 \Rightarrow y = \pm\frac{b}{a}x$

asymptotes

Why does it open "sideways"?

Let for a moment $a = b = 1$; then

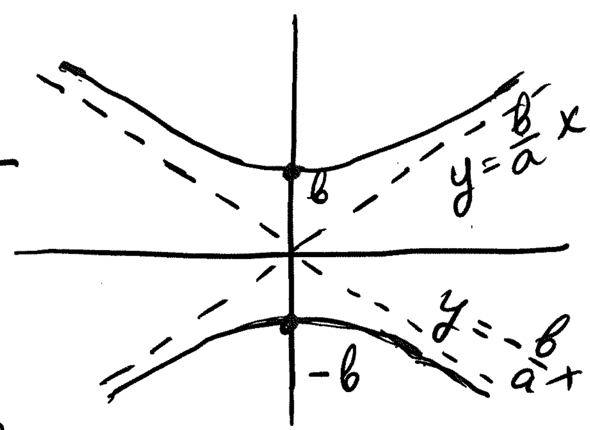
$x^2 - y^2 = 1$, i.e. $x^2 > y^2$.

Can take $y = 0$ (get the intercept $(\pm a, 0)$);

but cannot get $x = 0$: $0^2 - y^2 \neq 1$!

or

(**-II) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



$a = b = 1 \Rightarrow y^2 > x^2$

Opens up/down;

intercepts @ $(0, \pm b)$.

Parametric eqs. of hyperbolas:
see Lab 1, Part 1.

3 Cylindrical surfaces in 3D

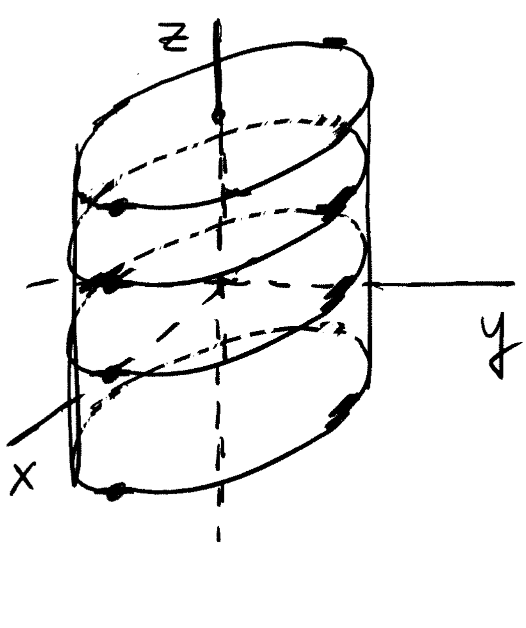
Consider $x^2 + y^2 = 1$ in 3D (not 2D).

Let's count degrees of freedom, as in Secs. 12.5A & 12.5B:

$$\underset{\substack{\uparrow \\ \text{\# of d.of f.} \\ \text{in } \mathbb{R}^3}}{3} - \underset{\substack{\nwarrow \\ \text{\# of} \\ \text{eqs.}}}{1} = 2$$

$\leftarrow$ So, the equation $x^2+y^2=1$ in 3D must define a surface, not a curve.

Since this equation does not envolve z, we can obtain this surface by drawing the circle $x^2+y^2=1$ in every plane $z=$ const:



We extrude the circle along the z-axis, obtaining a circular cylinder.

In general, any equation in 3D that is <u>missing a variable</u>, defines a <u>cylinder</u> obtained by extruding some 2D curve along the axis corresponding to the missing variable.

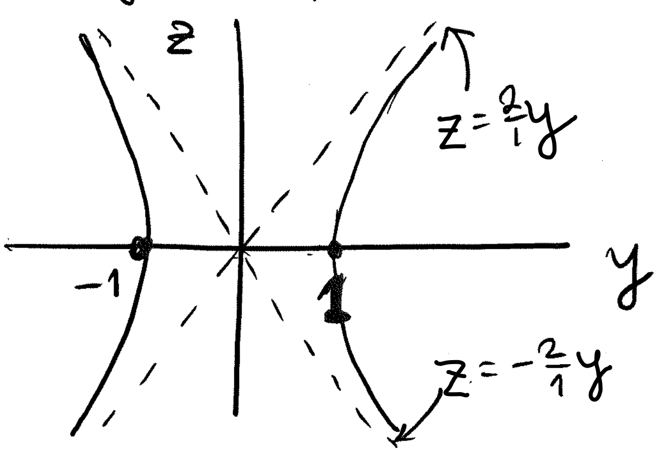
<u>Ex. 1(a)</u> What is $4y^2 - z^2 = 4$ in 3D?

<u>Sol'n:</u> 1) The equation is missing x. Therefore it is a cylinder obtained by extruding curve $\underbrace{(4y^2 - z^2 = 4,\ x = 0)}$ along the x-axis.

Note 2 eqs to denote a curve !!!

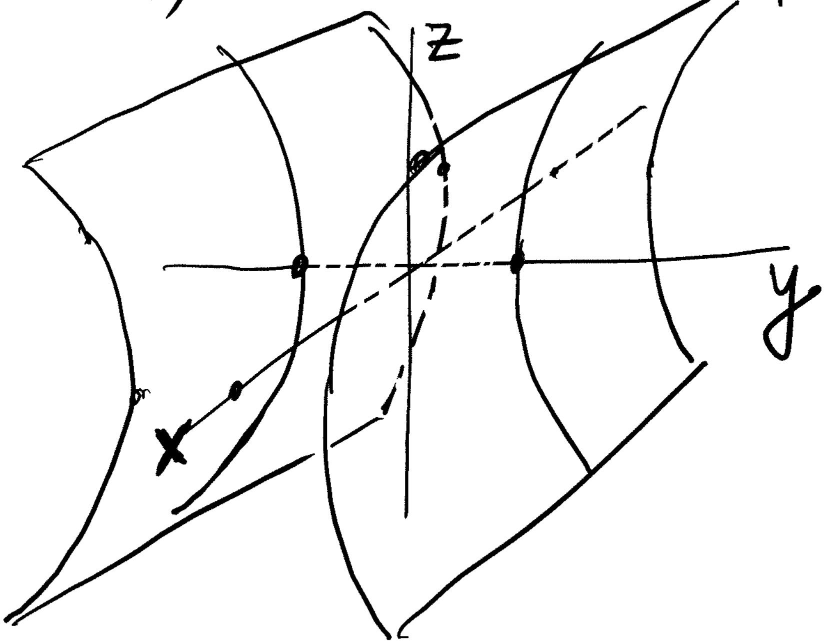
2) We now need to sketch the above curve. As on p. 6-2, must <u>put in canonical form</u>:

$$y^2 - \frac{z^2}{4} = 1.$$ This is a hyperbola.



y^2 ">" z^2 $\Rightarrow$ opens sideways (as in ~~(**)~~-I))

3) Sketch the surface:



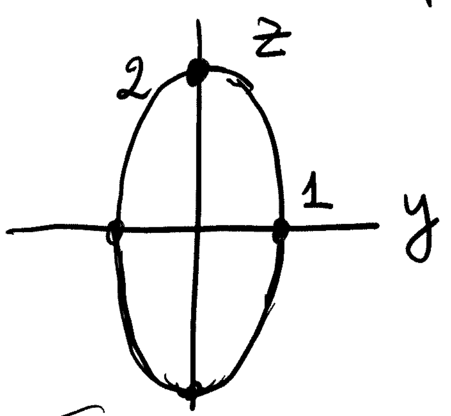
Hyperbolic cylinder

Ex. 1(b) What is $4y^2+z^2=4$ in 3D?

Sol'n: Follow the same steps.

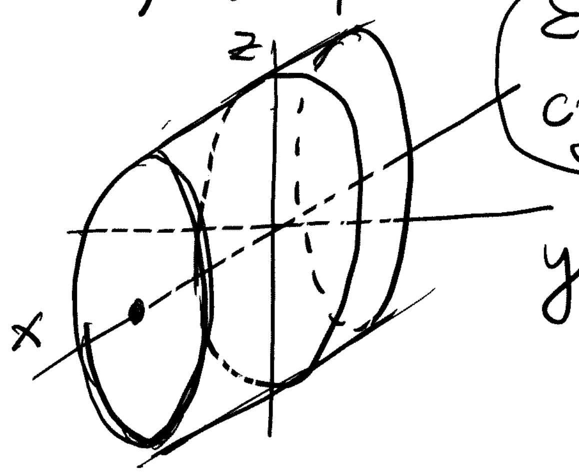
1) This must be some cylinder along the x-axis

2) Canonical form: $\frac{y^2}{1^2}+\frac{z^2}{2^2}=1$



ellipse

3) Surface:



Elliptical cylinder