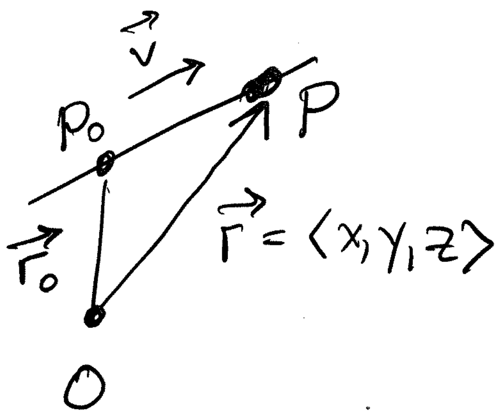
Sec. 13.1 Vector-valued functions

1 Parametric eqs. of curves in 3D

In Sec. 12.5A we learned that a line has the vector equation

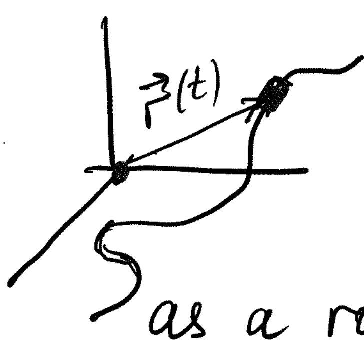


$$\vec{r}(t) = \vec{r}_0 + \vec{v}t,$$

or, equivalently, parametric equations

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct.$$

(You can think of these as the eqs. of motion of a car that starts to move at location $\vec{r}_0 = \vec{OP_0}$ and goes with a constant velocity $\vec{v} = \langle a, b, c \rangle$.)



Similarly, you can think of some general parametric function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ as a road in 3D, on which a car moves.

Thus, in a parametric curve with eqs

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

the radius vector

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

gives the location of a point on that curve.

This $\vec{r}(t)$ is called a <u>vector-valued function.</u>

Another notation:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

In 3D, a vector (-valued) function is an ordered list of 3 scalar-valued functions.

<u>Domain of a vector function $\vec{r}(t)$</u>: All values of t where all three components $x(t), y(t), z(t)$ are defined simultaneously. (See Ex. 1 in book)

(2) <u>Sketches of some vector-valued functions</u>

<u>Name</u>: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $\Rightarrow$
$x = f(t)$, $y = g(t)$, $z = h(t)$ defines a <u>parametric curve</u>.

<u>Ex. 1</u> Sketch a parametric curve
$$x = e^t, \quad y = \cos(2t), \quad z = \sin(2t).$$

<u>Sol'n</u>:

1) First, we recognize that in the projection on the yz-plane (i.e. if we <u>ignore</u> x-coord. for now), we have a circle (see Sec. 12.6)

$$y = \cos 2t$$
$$z = \sin 2t$$

<u>Q</u>: What interval of t gives a 360° turn?

2) What happens to x as t varies?,

$t=0 \Rightarrow x=e^0=1,\ y=\cos 0=1,\ z=\sin 0=0.$

$t \nearrow \Rightarrow x \nearrow$ exponentially to ∞.

$t \searrow \Rightarrow x=e^{-t}$; goes to 0.

The curve will wrap around the cylinder $y^2+z^2=1$ and will asymptotically approach the plane $x=0$, making infinitely many turns.

In the other direction along t, the curves moves away to $x=\infty$ while staying on the cylinder $y^2+z^2=1$.

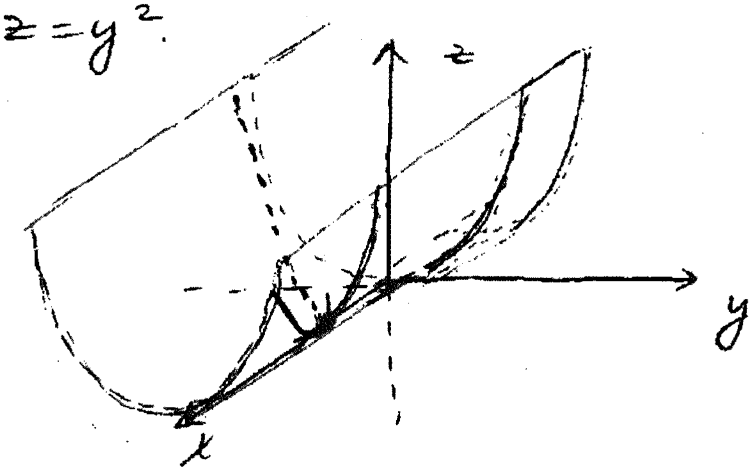
Some Helix's applications:
- DNA
- Electron with $\vec{v} \parallel \vec{B}$.

(Compare with Ex. 4 in book.)

MUST READ to learn about a HELIX

Ex. 2 Sketch the curve $\vec{r}(t)=\langle e^t, t, t^2\rangle$.

Sol'n: 1) In yz-plane we now have a parabola: $y=t,\ z=t^2,\ \Rightarrow z=y^2$:
So the curve will lie on the cylinder $z=y^2$.



2) As $t=0$: $x=e^0=1$, $y=0$, $z=0$.
$t \nearrow \Rightarrow x \nearrow$ exponentially.
$t \searrow \Rightarrow x = e^{-t} \to 0$

Note : The curve also belongs to the cylinder $x=e^t$, $y=t$, $\Rightarrow$ $x=e^y$, or $y=\ln x$.

Thus, ~~this~~ curve (and any other 3D curve!) is the intersection line of some two cylinders, e.g., $z=F(y)$ and $x=G(y)$. Thus, any line in 3D is defined by 2 eqs!

$$\underset{(x,y,z)}{3} - \underset{\#\text{eqs}}{2} = \underset{\text{remains degrees of freedom}}{1}$$

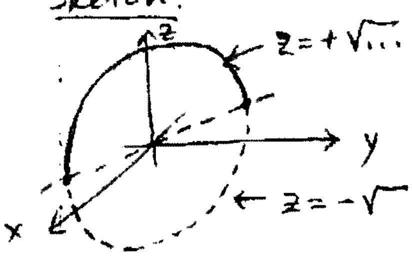
③ Parametric equations of intersection of lines & surfaces

(see Ex. 6 in book).

Ex. 3 Find the parametric eqs. of the curve obtained at the intersection of two given surfaces:
S_1: $x+y=0$ (plane), S_2: $x^2+y^2+z^2=2$

Sol'n: 1) Parametric eqs. are not unique. We can start by setting any of x, y, z equal to t.
E.g., let $x=t$.
2) S_1: $y=-x=-t$.
3) S_2: $z = \pm\sqrt{2-x^2-y^2} = \pm\sqrt{2-t^2-(t)^2}$
$= \pm\sqrt{2-2t^2} = \pm\sqrt{2}\sqrt{1-t^2}$.

Sketch:



Answer:
$x=t$
$y=-t$
$z=\pm\sqrt{2}\cdot\sqrt{1-t^2}$ (two Branches).

Note 1: We could have taken

$$y = t.$$

Then $x = -t$

$$z = \pm\sqrt{2-(-t)^2-t^2} = \pm\sqrt{2}\sqrt{1-t^2}.$$

So another parametric form of the same curve is

$$\langle -t, t, \pm\sqrt{2}\sqrt{1-t^2} \rangle.$$

Note 2: If we set $z = t$, we get:

$$x+y=0, \quad x^2+y^2+t^2 = 2$$

1st. eq: $y = -x$, $\Rightarrow$ $x^2+(-x^2)+t^2 = 2$

$$2x^2 = 2-t^2, \quad x = \pm\sqrt{1-\frac{t^2}{2}}$$

$$y = -x = \mp\sqrt{1-t^2/2}.$$

Curve: $\langle \pm\sqrt{1-t^2/2}, \mp\sqrt{1-t^2/2}, t \rangle$.

This is a somewhat more cumbersome form than above.

Moral: It pays off to pick the "most convenient" of x, y, z as t. (Or, if $x = t$ doesn't work easily, try $y = t$, etc.)

Ex. 4 ("inverse" to Ex. 3)

Show that the parametric curve $\begin{cases} x = t \\ y = -t \\ z = \sqrt{2-2t^2} \end{cases}$ lies on the sphere $x^2+y^2+z^2 = 2$.

Sol'n: All you need to do is to substitute the param. eqs. into the eq. of the surface and verify that it holds:

$$x^2 + y^2 + z^2 = 2$$

$$t^2 + (-t)^2 + (\sqrt{2-2t^2})^2 = 2$$

$$\cancel{t^2} + \cancel{t^2} + \cancel{2} - \cancel{2t^2} = \cancel{2} \qquad 0 = 0 \ \checkmark.$$

④ <u>Intersection of a parametric curve with a surface or another curve.</u>

Compare with: Ex. 6/book + Ex. 2/Notes for Sec. 12.5B.

<u>Ex. 5</u> At what points does the helix (see Ex. 4 in book)

$\vec{r}(t) = \cos t \cdot \vec{i} + \sin t \, \vec{j} + t \vec{k}$

intersect the sphere $x^2 + y^2 + z^2 = 4$?

<u>Sol'n:</u> All you need to do is to substitute $x(t), y(t), z(t)$ into the eq. of the surface and find for what t the eq. is true:

$$x^2 + y^2 + z^2 = 4$$

$$\underbrace{(\cos t)^2 + (\sin t)^2}_{1} + (t)^2 = 4$$

$$1 + t^2 = 4 \implies t = \pm\sqrt{3}.$$

<u>Answer:</u> $\underline{t = -\sqrt{3}}$: $P_1 = (\cos(-\sqrt{3}), \sin(-\sqrt{3}), -\sqrt{3})$

$t = \sqrt{3}$: $P_2 = (\cos\sqrt{3}, \sin\sqrt{3}, \sqrt{3})$.

<u>Discuss:</u> Intersection of two curves

$\vec{r}_1(t) = \langle f_1(t), g_1(t), h_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t), h_2(t) \rangle$:

For what t and s is $\vec{r}_1(t) = \vec{r}_2(s)$?