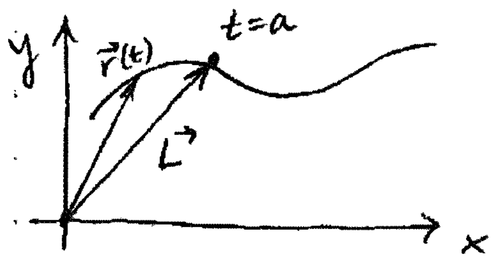
Sec. 13.2. Derivatives & integrals of vector functions

① Limits and continuity of v.f.

Draw all pictures in 2D; conclusions apply in 3D.

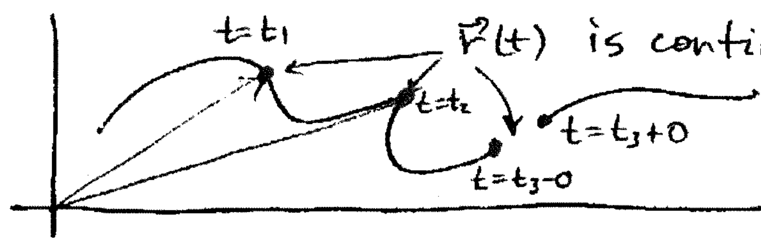


$$\lim_{t\to a} \vec{r}(t) = \vec{L}$$

if $\vec{r}(t)$ approaches $\vec{L}$ as $t \to a$.

" $\vec{r}(t) \longrightarrow \vec{L}$ " means "each component of $\vec{r}$ $\to$ the corresponding component of $\vec{L}$".

V.f. $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at $t = t_0$ if the param. curve $\vec{r} = \vec{r}(t)$ $(x = f(t),\ y = g(t),\ z = h(t))$ has no jumps at $t = t_0$.



$\vec{r}(t)$ is continuous at t_1, t_2, but discontinuous at $t = t_3$.

② Derivatives of a v.f.

Meaning:

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle \leftarrow$ location (position)

$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle \leftarrow$ velocity

$\vec{r}\,''(t) = \langle f''(t), g''(t), h''(t)\rangle \leftarrow$ acceleration

Differentiation rules (p. 858, 8th Ed.)

1) $(\vec{u} \pm \vec{v})' = \vec{u}' \pm \vec{v}'$

2) $c = \text{const} \Rightarrow (c \cdot \vec{u})' = c\vec{u}'$

$(\vec{c} = \text{const vector}) \Leftrightarrow \vec{c}' = \vec{0}$

3) $(f(t) \cdot \vec{u})' = f'\vec{u} + f \cdot \vec{u}'$

4) $(\vec{u} \bullet \vec{v})' = \vec{u}' \bullet \vec{v} + \vec{u} \bullet \vec{v}'$

$(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

Product Rules

E.g., here is a Proof of 3) (see book for 4)):

$$(f \cdot \vec{u})' \equiv (f \cdot \langle u_1, u_2 \rangle)' = \langle f u_1, f \cdot u_2 \rangle' =$$

$$= \langle (f \cdot u_1)', (f \cdot u_2)' \rangle =$$

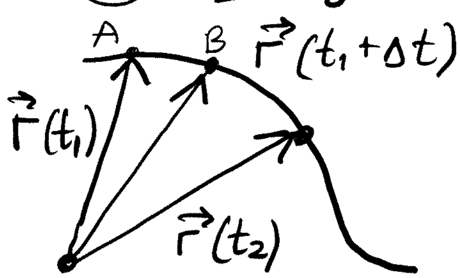
$$= \langle \underline{f' u_1} + \underline{\underline{f u_1'}}, \underline{f' u_2} + \underline{\underline{f u_2'}} \rangle =$$

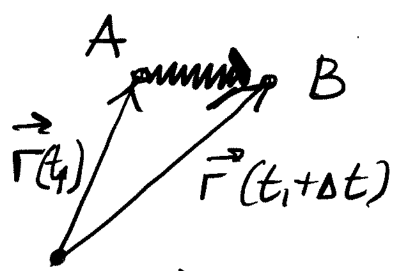
$$= \langle f' u_1, f' u_2 \rangle + \langle f \cdot u_1', f u_2' \rangle$$

$$= f'\vec{u} + f\vec{u}'.$$

//

③ Tangent vector to a curve





$$\vec{r}(t_1) + \vec{AB} = \vec{r}(t_1 + \Delta t)$$

$$\vec{AB} = \vec{r}(t_1 + \Delta t) - \vec{r}(t_1) \quad \text{(see Sec. 12.2!)}$$

As $\Delta t \to 0$, $\vec{AB} \to \vec{0}$, but $\lim\limits_{\Delta t \to 0} \dfrac{\vec{AB}}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \vec{r}'(t)$.

$\boxed{\text{Thus, } \vec{r}'(t_1) \text{ is always tangent to } \vec{r}(t) \text{ @ } t = t_1.}$

MUST READ/FOLLOW THE PROOF OF Theorem 4 in **textbook**.

Discussion of meaning of Theorem 4:

"A bug is crawling on a sphere centered at the origin. Show that the bug's velocity is always $\perp$ to its position vector $\vec{r}(t)$."

Ingredients of solution:

Given: Sphere $\Rightarrow$ $|\vec{r}(t)| = a$ ← some const

Want: velocity $\perp \vec{r}(t) \iff$

$\vec{r}\,' \perp \vec{r}$.

Steps of solution (helping you to go through Thm. 4)

- Relate the length $|\vec{r}|$ to some dot product (See Notes on Sec. 12.3).
- Use the Product Rule for the dot product (see previous page).
- Use properties of the dot product (Sec. 12.3) after you differentiate.
- Finally, from your calculation, use a criterion from Sec. 12.3 that shows that two vectors are $\perp$. //

Ex. 1 Find a unit tangent vector to $\vec{r} = \langle \sin 3t, \cos^2 t \rangle$ at $t = \pi/3$.

Sol'n: 1) A tangent vector is $\vec{r}\,'(t) = \langle \sin 3t, \cos^2 t \rangle' = \langle (\sin 3t)', (\cos^2 t)' \rangle$

Absolutely MUST DO!

!

Chain Rule reminder [Review Sec. 3.4 on your own!]

$$\frac{d}{dt}(\sin 3t) = \frac{d}{dt} \sin u(t) = \frac{d \sin u}{du} \cdot \frac{du}{dt} \qquad u(t) = 3t$$

$$= \cos u \cdot \frac{d(3t)}{dt} = \cos(3t) \cdot 3$$

$$= 3 \cos 3t.$$

$$\frac{d(\cos^2 t)}{dt} = \frac{d\, u^2(t)}{dt} = \frac{du^2}{du} \cdot \frac{du}{dt} \qquad \cos t = u$$

$$= 2u \cdot \frac{d(\cos t)}{dt} = 2\cos t \cdot (-\sin t) = -2\cos t \cdot \sin t.$$

Thus, $\vec{r}\,'(t) = \langle 3\cos 3t, -2\cos t \cdot \sin t \rangle$

2) Unit tan. vector @ $t = t_0 = \pi/3$:

$$\vec{r}\,'(t_0) = \langle 3\cos 3t_0, -2\cos t_0 \sin t_0 \rangle$$

$$= \langle 3\underbrace{\cos \pi}_{-1}, -2\underbrace{\cos \tfrac{\pi}{3}}_{1/2} \cdot \underbrace{\sin \tfrac{\pi}{3}}_{\sqrt{3}/2} \rangle = \langle -3, -\sqrt{3}/2 \rangle.$$

Recall from Ex. 4 in Notes for Sec. 12.2 that any vector $\vec{u}_* = \dfrac{\vec{u}}{|\vec{u}|}$ has unit length.

Thus, $\dfrac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$ is a unit tangent vector to curve $\vec{r} = \vec{r}(t)$ at $t = t_0$.

$|\vec{r}'(t_0)| = \sqrt{(-3)^2 + (-\sqrt{3}/2)^2} = \sqrt{39/4}$.

Answer: $-\dfrac{\langle 3, \sqrt{3}/2 \rangle}{\sqrt{39/4}}$.

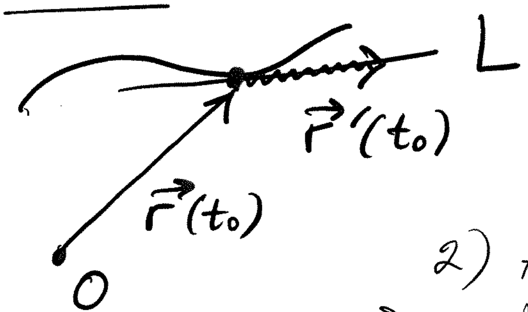
Generalization:

$$\boxed{\vec{T}(t) \equiv \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \text{ is the unit tangent vector to curve } \vec{r} = \vec{r}(t).}$$

Ex. 2 (see also Ex. 3 in book)

Find the parametric eqs. for the tangent line to the curve of Ex. 1 at $t_0 = \pi/3$.

Sol'n:



1) We need 2 ingredients of a line:
- A point on it;
- A vector along it.

2) A point on it is

$\vec{r}(t_0)\Big|_{t_0 = \pi/3} = \langle \sin(3 \cdot \frac{\pi}{3}), \cos^2 \frac{\pi}{3} \rangle = \langle 0, 1/4 \rangle$.

3) $\vec{r}'(t_0)\Big|_{t_0 = \pi/3} \overset{\text{Ex. 1}}{=} \langle -3, -\sqrt{3}/2 \rangle$

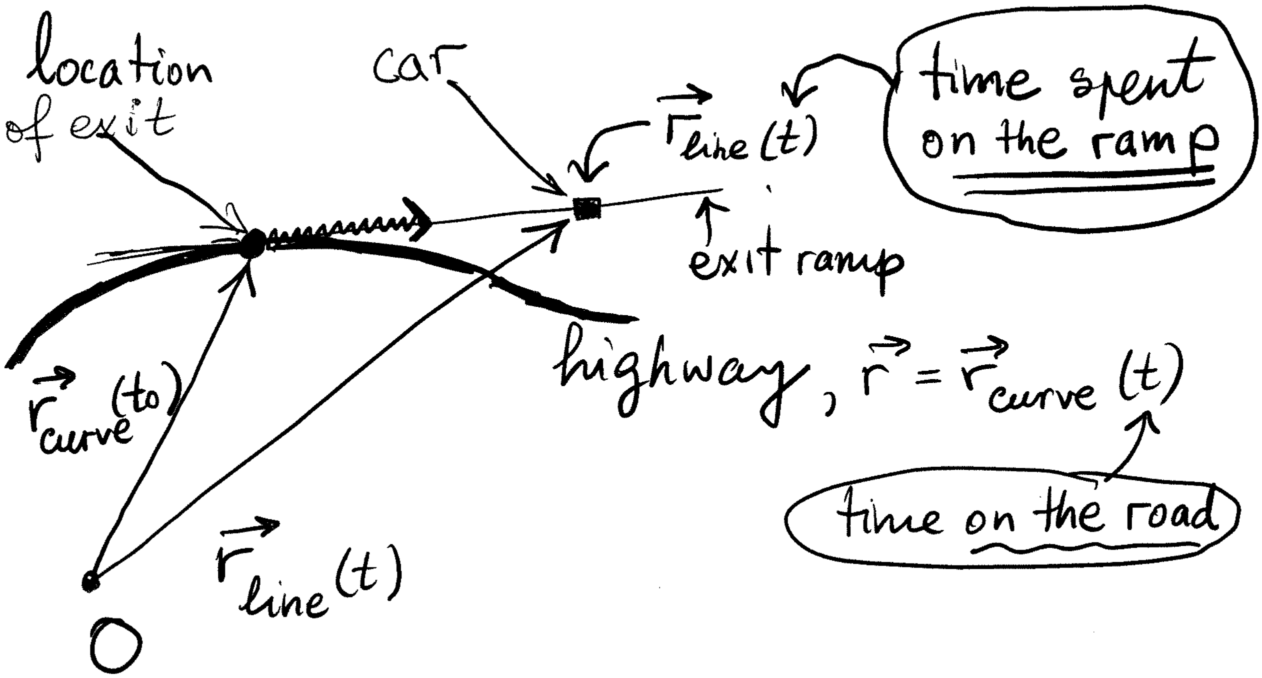
Note: We don't need to make it a unit vector.

4) Vector eq. of the tan. line: $\vec{r}_{line}(t) = \vec{r}_{curve}(t_0) + \vec{r}'_{curve}(t_0) \cdot t$

Parametric eqs: $\langle x, y \rangle_{line} = \langle 0, 1/4 \rangle + \langle -3, -\frac{\sqrt{3}}{2} \rangle \cdot t$.

Highway-exit analogy for the tangent line to a curve:

car exits at $t = t_0$ (time measured on the road)

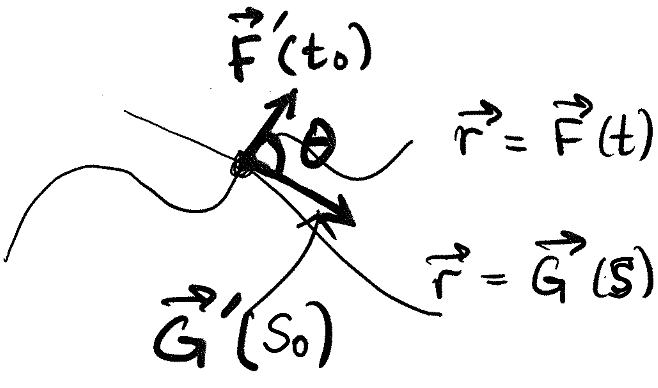


time on exit ramp

$$\underbrace{\vec r_{line}(t)}_{\text{location on exit ramp}} = \underbrace{\vec r_{curve}(t_0)}_{\text{location of exit}} + \underbrace{\vec r\,'_{curve}(t_0)}_{\text{velocity at exit}} \cdot t$$

time when the car arrives at the exit on highway

Note about the angle between two intersecting curves:



If curves $\vec{r} = \vec{F}(t)$ and $\vec{r} = \vec{G}(t)$ intersect at some point where $\vec{F}(t_0) = \vec{G}(s_0)$, then the angle between them is defined as the angle between their tangent vectors at the intersection point. See Sec. 12.3.

4 Integrals of vector functions

Idea: Simply integrate each component of the vector function (see Ex. 5 in the book).

Ex. 3 Find $\vec{r}(t)$ if $\vec{r}\,'(t) = \vec{i}\cdot\cos t + \vec{j}\cdot\sin t$ and $\vec{r}(\pi/2) = 3\vec{i} - \vec{j}$.

Interpretation: Find the location of a car ($\vec{r}(t)$) at all times if you know its velocity ($\vec{r}\,'(t)$) and location at some instance of time ($t_0 = \pi/2$).

8-8

Sol'n:

1) $\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \langle \cos t, \sin t \rangle dt =$

$= \langle \int \cos t\,dt, \int \sin t\,dt \rangle = \langle \sin t + C_1, -\cos t + C_2 \rangle.$

Note that the constants for each component are <u>different</u>.

2) Find C_1, C_2.

At $t = \pi/2$, $\vec{r}(\pi/2) = \langle \sin\frac{\pi}{2} + C_1, -\cos\frac{\pi}{2} + C_2 \rangle$

$= \langle 1 + C_1, C_2 \rangle.$

On the other hand, we are <u>given</u> that $\vec{r}(\pi/2) = \langle 3, -1 \rangle$. Thus:

$\langle 1 + C_1, C_2 \rangle = \langle 3, -1 \rangle \Rightarrow$

$$\begin{cases} 1 + C_1 = 3 \Rightarrow C_1 = 2 \\ C_2 = -1 \end{cases}$$

<u>Answer:</u> $\vec{r}(t) = \langle (\sin t) + 2, (-\cos t) - 1 \rangle$.