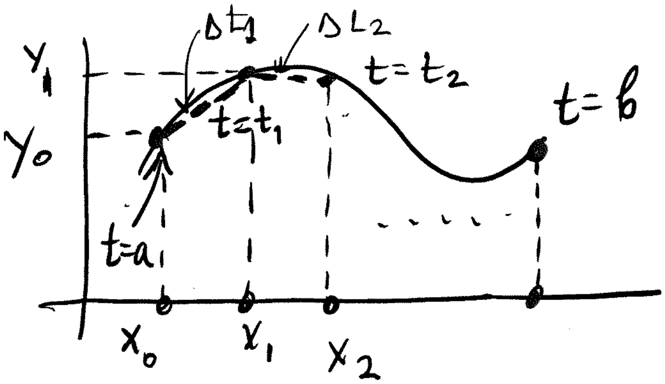
Sec. 13.3 Arclength & curvature

① Arclength



The following is a reminder of the arclength derivation in 2D from Sec. 10.2. Approximate the length L of the smooth curve by the sum of lengths of straight-line segments:

$$L \approx \sum_i (\Delta L)_i = \sum_i \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} \equiv \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_i \sqrt{\Delta t^2 \left(\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 \right)}, \quad \text{where } \boxed{\Delta t = t_{i+1} - t_i}$$

But $\dfrac{\Delta x_i}{\Delta t} \equiv \dfrac{x(t_i+\Delta t) - x(t_i)}{\Delta t} \approx \dfrac{dx}{dt}$, so as $\Delta t \to 0$,

$$L \approx \sum_i \Delta t \cdot \sqrt{\left(\frac{dx}{dt}\right)_i^2 + \left(\frac{dy}{dt}\right)_i^2} \equiv \sum_i \Delta t \underbrace{\sqrt{(x_i')^2 + (y_i')^2}}_{|\langle x_i', y_i' \rangle|}$$

$$\xrightarrow[\Delta t \to 0]{} \int_{t=a}^{t=b} |\langle x'(t), y'(t) \rangle| \, dt$$

Now recall: $\vec{r}(t) \equiv \langle x(t), y(t) \rangle$ (we're in 2D),

$$\Rightarrow \vec{r}\,'(t) = \langle x'(t), y'(t) \rangle.$$

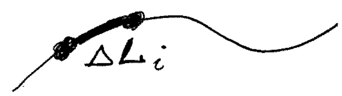
Thus:

$$\boxed{L = \int_{t=a}^{t=b} |\vec{r}\,'(t)| \, dt} \qquad (*)$$

See Ex. 1 / book (Sec. 13.3) for numbers
(remember that you had to read Ex. 4/Sec. 13.1 about a helix).

Discussion of formula (*).

- Recall that $\vec{r}\,'(t)$ is the "velocity of a car on the road".
  Then $|\vec{r}\,'(t)|$ is the speed of the car.
  Velocity is a vector : has magnitude & direction.
  Speed is a scalar : has only magnitude (≥ 0).
- $$L \approx \sum_i \underbrace{\Delta L_i}_{\sqrt{\Delta x_i^2 + \Delta y_i^2}\,\cdot\,\Delta t \;\equiv\; |\vec{r}\,'(t_i)|\,\Delta t}$$



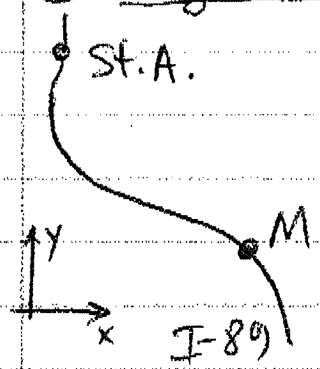
$$\equiv \sum_i \underbrace{(\text{speed at time interval } i)\cdot \Delta t}_{\text{distance traveled during this time interval}}$$

So
$$L = \int_{t=a}^{t=b} |\vec{r}\,'(t)|\, dt$$

can be interpreted as:

$$\begin{pmatrix}\text{traveled}\\ \text{distance}\end{pmatrix} = \int_{t=a}^{t=b} \text{speed}(t)\cdot dt$$

(2) Using arc length as a parameter



Consider a section of a highway. We can follow any given car, measuring its $x(t)$, $y(t)$, where t is time. Then

$$x = x(t), \quad y = y(t)$$

will be the parametric eqs. of this interstate section.

They will ~~depend~~ can be different for different cars (because cars go at different speeds). So the same curve can be parametrized in many different ways.

<u>Q</u>: Is there a way to parametrize this curve which will <u>not</u> depend on properties of a particular car + driver, but will depend <u>only</u> on the properties of the curve?

<u>A</u>: Yes: Use s, the travelled distance, as the parameter.

E.g., at $s = 10$ miles north of Montpelier along I-89, $x(s=10) = x_0$, $y(s=10) = y_0$.

<u>Step 1</u>: Introduce arc length function:

$$s = \int_a^t |\vec{r}\,'(\tilde{t})|\, d\tilde{t} \quad \leftarrow \text{distance travelled by a particular car}$$

$\Rightarrow$ get a function $s = F(t)$ for some F.

Step 2: Solve for $t = F^{-1}(s)$.

Step 3: Substitute into $\vec{r}(t) = \vec{r}(F^{-1}(s))$, which then becomes a function of s:

$$\boxed{\vec{r} = \vec{r}(s)} \leftarrow \text{arc length parametrization of the given curve.}$$

Ex. 1 Find the arc length parametrization of a helix ← (Ex. 4 / Book for Sec 13.1)

$$\vec{r}(t) = 3\sin t \cdot \vec{i} + 4t\vec{j} - 3\cos t\, \vec{k},$$

when the arc length is measured from point $\langle 0, 4\pi, 3\rangle$ in the direction of increasing t.

Sol'n:

Step 0: Find $t=a$: the initial point $(0, 4\pi, 3)$:

$$\begin{cases} 3\sin a = 0 \\ 4a = 4\pi \\ -3\cos a = 3 \end{cases} \Rightarrow a = \pi.$$

Step 1: $s = \int_{\pi}^{t} |\vec{r}'(\tilde{t})|\, d\tilde{t}$

$$|\vec{r}'(\tilde{t})| = \sqrt{(3\cos\tilde{t})^2 + 4^2 + (3\sin\tilde{t})^2} = 5.$$

$$s = \int_{\pi}^{t} 5\, d\tilde{t} = 5(t-\pi).$$

Step 2: Solve for t: $t = \frac{s}{5} + \pi$.

Step 3: Sub. into $\vec{r}(t)$:

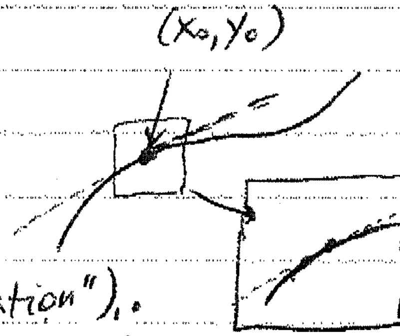
$$\vec{r}(s) = \left\langle 3\sin\left(\pi + \frac{s}{5}\right), 4\left(\frac{s}{5}+\pi\right), -3\cos\left(\pi + \frac{s}{5}\right)\right\rangle$$

$$= \left\langle -3\sin\left(\frac{s}{5}\right), \frac{4}{5}s + 4\pi, 3\cos\left(\frac{s}{5}\right)\right\rangle.$$

(3) Motivation for the remainder of this section.

Essense of Calculus

In 2D: 1) Each smooth curve is *locally* ~~app~~ approximated by its tangent line.



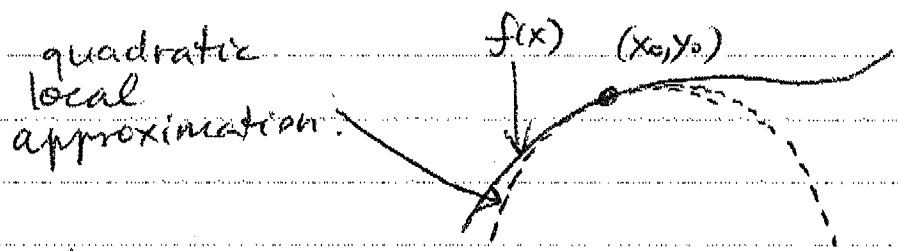
(See Sec. 3.10, "tan. line approximation").

So: "Locally, any smooth curve is **just** a line."

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

2) If the line is curved, we can improve the approximation using a truncated Taylor series (quadratic approximation):

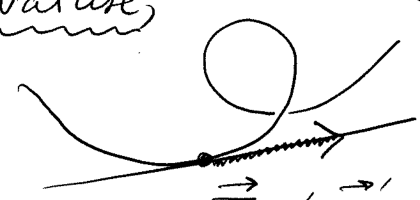
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$



So, we can capture the *curvature* of the curve by adding the term $\sim f''(x_0)(x - x_0)^2$.

Q: How does all this generalize to 3D?

Answer: 1) If one ignores curvature, then locally (i.e. close to any given point), a smooth curve can be approximated by the tangent line at that point (See Sec. 13.2).

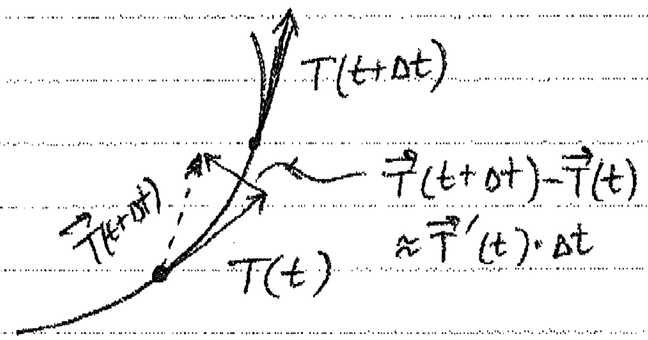


2) What if one wants to include curvature? Is there some curve that yields a "local quadratic approximation" to a given curve?
To answer this, we will follow the

Plan:

Step 1: Accept as a fact that locally (i.e., again, near any given point), our curve approximately lies in some plane. (We will later show how to find that plane.)
Step 2: We will postulate the shape of a curve that will serve as "the local quadratic approximation" (like a parabola was the loc. quadr. approx. in Calculus II). This will now be a circle.
And then we'll show how to compute the radius of the circle and, finally, describe its relation to curvature.

We have accepted that any curve in 3D can be viewed <u>locally</u> as lying in a plane. So consider a 2D curve.



Let us show that
$\vec{T}'(t) \perp \vec{T}(t)$
for all t.
Then $\vec{T}'$ can be chosen as the direction of the normal vector to the curve.

<u>Know:</u>

$$|\vec{T}| = 1$$

$$|\vec{T}|^2 = 1$$

$$(\vec{T}\cdot\vec{T} = 1)'$$

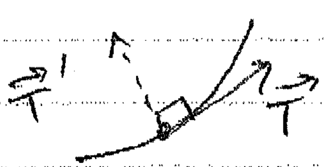
(Sec. 13.2; Thm.4/book)

$$\vec{T}'\cdot\vec{T} + \vec{T}\cdot\vec{T}' = 0$$

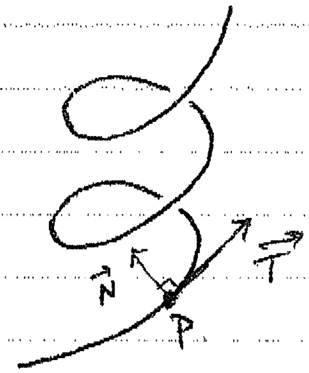
$$2\vec{T}'\cdot\vec{T} = 0 \Rightarrow \vec{T}' \perp \vec{T}.$$

Then $\boxed{\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}}$ is the <u>unit</u> normal vector

<u>Note:</u> It points "inside" the curve (see the figure above). + below.



See Ex. 6 in book for finding $\vec{N}(t)$ for a helix.



Osculating plane

- At each pt. P of a smooth curve we can find its tangent and normal vectors.
- Two vectors ($\vec{T}$ & $\vec{N}$ in this case) define a plane where they lie.
- This plane comes the closest to containing the part of the curve near P. (This is what we asked in Sub-Q 2.2). So, locally, the curve near P approximately lies within a plane.
- This plane is called the osculating plane at pt. P.

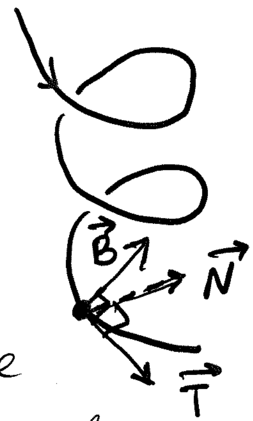
To compute the eq. of the osculating plane, one has: 1) pt. P
2) two vectors $\vec{T}$ & $\vec{N}$ in that plane.

Also topic 3 12.5B (notes)

This problem was solved in Sec. 12.5 (in book see Ex. 5 for Sec. 12.5; See also Ex. 7 for Sec. 13.3). (book)

Normal plane: A plane through pt. P and $\perp$ to $\vec{T}$. (Ex. 7 in book).

Binormal vector



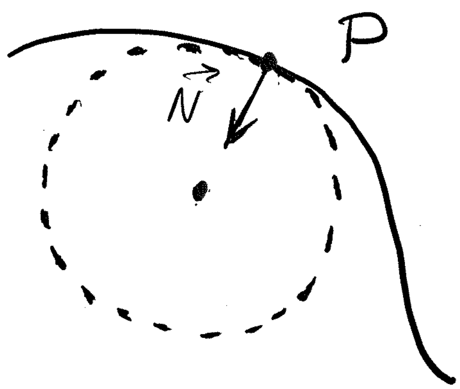
A child is sliding down a helical slide. At each point, she has her "local" x- and y-axes: her legs (for x-axis) and (left) hand (for the y-axis). Then her trunk makes a "local z-axis". It points along a unit vector

$$\vec{B} = \vec{T} \times \vec{N},$$

where $\vec{B} \perp \vec{T}$ and $\vec{B} \perp \vec{N}$ by Sec. 12.4.
This is the "local" analog of $\vec{k} = \vec{i} \times \vec{j}$.

5 Curvature

(Step 2 of the Plan on p. 9-5)

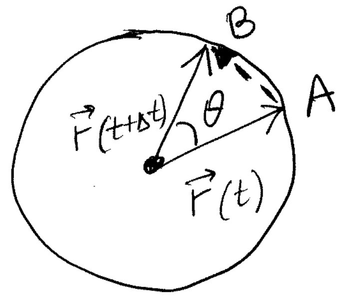


Consider an arbitrary curve and a point P on it. Intuitively, there is a unique circle that fits the curve most closely near P.

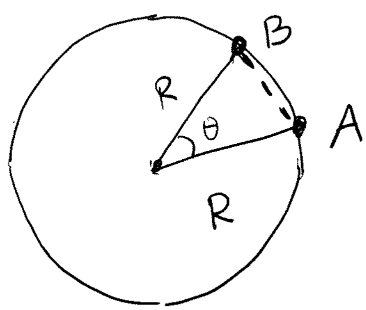
It is called the osculating circle to the curve at point P. It replaces the parabola (p. 9-4) as the "local quadratic approximation" for 3D curves.

We will now show how to find the radius of the osculating circle.

The *idea* is to pretend that while the



point stays on the osculating circle, its equation is given by the equation $\vec{r} = \vec{r}(t)$ of the curve. (This is justified by the closeness of the circle to the curve.)



So, we are going to compute R (radius of the circle) from the known vector function $\vec{r}(t)$.

Preliminary facts / reminders:

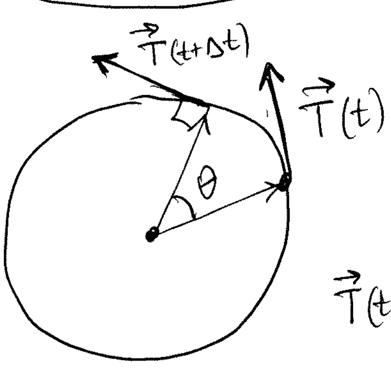
- $\breve{AB} = R \cdot \theta$ (see picture above).
  E.g., $\theta = 2\pi \Rightarrow$ circumference $= 2\pi \cdot R$.
- If θ is small, then $\breve{AB} \approx |\vec{AB}|$.
  Thus, (θ is small) $\Rightarrow \boxed{|\vec{AB}| \approx R \cdot \theta}$ //

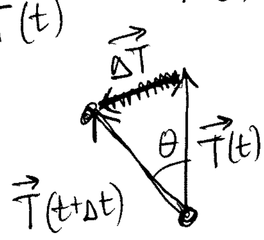
Now, let's look at the top picture on this page:

$$|\vec{AB}| = |\vec{r}(t+\Delta t) - \vec{r}(t)| \equiv |\Delta \vec{r}|.$$

Sec. 12.2

Thus, $|\Delta \vec{r}| \approx R \cdot \theta$





Next, consider the change in the unit tangent vector. Similarly to the above,

$$|\Delta\vec{T}| \approx |\vec{T}|\cdot\theta = 1\cdot\theta$$

From the last two circled eqs. we can eliminate θ:

$$\frac{|\Delta\vec{r}|}{|\Delta\vec{T}|} = \frac{R\cdot\cancel{\theta}}{1\cdot\cancel{\theta}} = R.$$

We are not done yet since we don't know what $\Delta\vec{r}$ and $\Delta\vec{T}$ are. But since they occur during a time interval Δt, let's divide both the numerator and denominator by Δt:

$$R = \frac{|\Delta\vec{r}/\Delta t|}{|\Delta\vec{T}/\Delta t|} \xrightarrow[\Delta t\to 0]{} \frac{|\vec{r}\,'(t)|}{|\vec{T}\,'(t)|}$$

(see p. 9-1)

Conventionally, one defines the <u>curvature</u> as $\frac{1}{R}$.

Why $1/\ldots$?

The larger the R, the less curved the curve.

" smaller ——— more curved ———.

(What is the curvature, and what is R, of a straight line?)

See Ex. 4/book for numbers.

Thus:

difficult to use

Easy to use

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$$

notation for curvature (Greek letter "kappa")

derived in book

Curvature of a curve $y = f(x)$:

- Parametrize $x = t$, $y = f(t)$, $z = 0$;
- Apply the general formula for $\kappa(t)$.

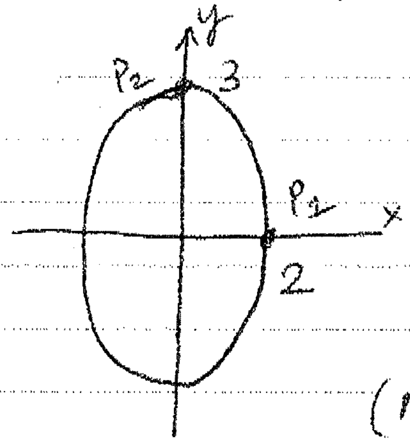
See Ex. 5 in the book.

Ex. 2 Find the curvature of the ellipse $\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$ at $P_1 = (2,0)$ & $P_2 = (0,3)$.

Sol'n:

0) The equation is given to us in 2D. Normally, this is ok to just use it as given. But when computing $\kappa(t)$, we need to compute the cross-product $(\vec{r}\,' \times \vec{r}\,'')$, which we only know how to do in 3D (Sec. 12.4). Therefore, even though we will make a sketch of the ellipse in 2D, we need to restate $\vec{r}(t)$ in 3D to be able to use cross product. So:

$$\vec{r}(t) = \langle 2\cos t, 3\sin t, 0 \rangle.$$



1) $K(t) = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$

$\vec{r}\,' = \langle -2\sin t, 3\cos t, 0 \rangle$

$\vec{r}\,'' = \langle -2\cos t, -3\sin t, 0 \rangle$.

$\vec{r}\,' \times \vec{r}\,'' = 6\vec{k}$

(Note: Since $\vec{r}\,'$, $\vec{r}\,''$ are in the xy-plane, their cross product must be $\perp$ to xy-plane, i.e. is along z-axis.)

$$K(t) = \frac{|6\vec{k}|}{\left(\sqrt{(-2\sin t)^2 + (3\cos t)^2 + 0}\right)^3} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}$$

2) Find t for P_1 and P_2:

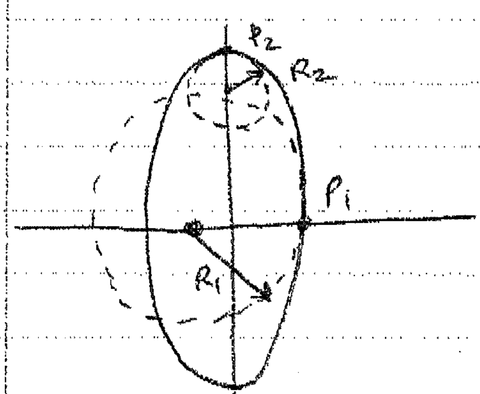
P_1: $(2,0,0) = (2\cos t_1, 3\sin t_1, 0) \Rightarrow t_1 = 0$.

P_2: $(0,3,0) = (2\cos t_2, 3\sin t_2, 0) \Rightarrow t_2 = \pi/2$.

@ P_1: $K(t_1) = \dfrac{6}{(4\cdot 0 + 9\cdot 1)^{3/2}} = \dfrac{6}{27} = \dfrac{2}{9}$

@ P_2 $K(t_2) = \dfrac{6}{(4\cdot 1 + 9\cdot 0)^{3/2}} = \dfrac{6}{8} = \dfrac{3}{4}$

$$K|_{P_1} < K|_{P_2} \Rightarrow R|_{P_1} > R|_{P_2}$$



(See a better picture in the Answer to #55 for Sec. 13.3 — not assigned as a HW problem.)