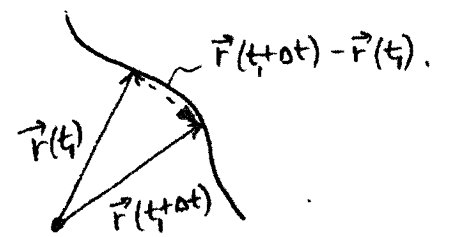
Sec. 13.4. Motion along a curve; velocity & acceleration

① Velocity, acceleration, and speed.

Note: Most of this was already covered in Sec. 13.2, so this is just a reminder.



Suppose the particle moves along a curve $\vec{r} = \vec{r}(t)$, Between two nearby moments $t=t_1$ and $t=t_1+\Delta t$ its <u>displacement</u> is

$\Delta\vec{r}(t_1) = \vec{r}(t_1+\Delta t) - \vec{r}(t_1)$.

So the <u>average</u> velocity during this time interval is $\vec{v}_{ave}(t_1) = \dfrac{\Delta\vec{r}(t_1)}{\Delta t}$,

and as $\Delta t \to 0$:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \equiv \vec{r}\,'(t);$$ and $\vec{v}$ is tangent to curve.

$$\boxed{\vec{v}(t) = \vec{r}\,'(t)}$$

Velocity. It is a vector; has magnitude, $|\vec{v}|$, and direction.

Magnitute of velocity is called <u>speed</u>; it's a number, not vector.

$$\boxed{\text{Speed} = |\vec{v}| \equiv |\vec{r}\,'(t)|}$$

(Recall: $\vec{v}$ – velocity, vector;
$|\vec{v}|$ – speed, scalar)

$\boxed{\text{Similarly, acceleration is the rate of change of the velocity:}}$

$$\boxed{\vec{a}(t) = \vec{v}\,'(t) = (\vec{r}\,'(t))' = \vec{r}\,''(t)}$$

Ex. 1 The position of a particle is given by

$$\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle.$$

Find its velocity, speed, and acceleration. Also find $|\vec{a}|$.

Sol'n

1) $\vec{v} = \vec{r}\,' = \langle R(\cos(\omega t))', R(\sin(\omega t))'\rangle$

Chain Rule $\longrightarrow \overset{\text{}}{=} \langle -R\omega\sin\omega t, R\omega\cos\omega t\rangle$

2) $|\vec{v}| = \sqrt{(-R\omega\cdot\sin\omega t)^2 + (R\omega\cos\omega t)^2}$

$= \sqrt{(R\omega)^2\sin^2\omega t + (R\omega)\cos^2\omega t}$

$= \sqrt{(R\omega)^2(\sin^2\omega t + \cos^2\omega t)} = \sqrt{(R\omega)^2}\sqrt{\sin^2\omega t + \cos^2\omega t} = R\omega.$

3) $\vec{a} = \vec{v}\,' = \langle -R\omega\cdot\omega\cos\omega t, -R\omega\cdot\omega\sin\omega t\rangle$

$= -\omega^2\langle R\cos\omega t, R\sin\omega t\rangle = -\omega^2\vec{r}.$

4) $|\vec{a}| = \omega^2\cdot|\vec{r}| = \omega^2\sqrt{R^2\cos^2\omega t + R^2\sin^2\omega t} \overset{\text{similar to 2)}}{=} \omega^2 R.$

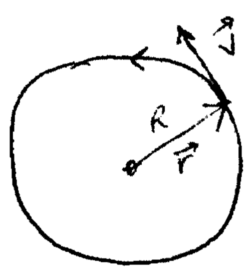
Discussion:

0) The curve is a circle with rad. R (recognized by its param. equation)

When time increases by T s.t. $\omega T = 2\pi$:

$\cos\omega(t+T) = \cos(\omega t + \omega T) = \cos(\omega t + 2\pi) = \cos\omega t$

similar for sin.



So time T s.t. $\omega T = 2\pi$, or $T = 2\pi/\omega$, is the period of revolution.

1) $\vec{v}$ is tangent to circle (and hence $\perp \vec{r}$ at every point of the circle):

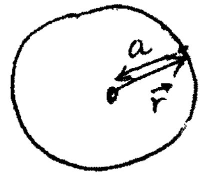
$$\vec{r} \perp \vec{v} \; (= \vec{r}\,').$$

This was proven earlier:

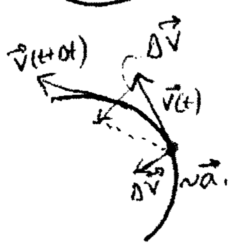
$$\left.\begin{array}{l} \text{(Sec. 13.2)} = \text{Thm.4/book} \\ \text{\& derivation of } \vec{N} \text{ in Sec. 13.3} \end{array}\right\} \quad |\vec{r}\,| = \text{const} \Rightarrow \vec{r} \perp \vec{r}\,'.$$

2) $|\vec{v}| = R\omega$ ← velocity of uniform circular motion

3) $\vec{a} = -\omega^2 \vec{r}$



In uniform circular motion, acceleration is directed towards the center of the circle. (centripetal acceleration)



curvature

4) $|\vec{a}| = \omega^2 R = \omega^2 R^2/R = |\vec{v}|^2/R \equiv |\vec{v}|^2 \kappa$ (Sec. 13.3)

magnitude of accel. of uniform circ. motion.

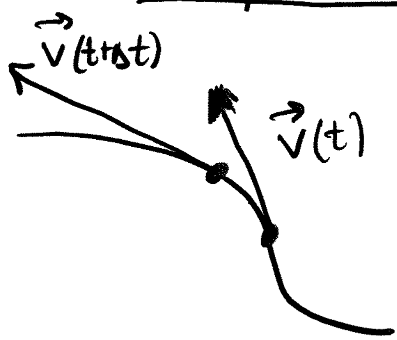
Will use all these facts now, as we move on from the uniform motion on a circle

osculating circle (Sec. 13.3)

to the general motion on any curve. Motivation: any smooth curve is locally approximated by an osculating circle.

Go to next page

2 Normal and tangential components of acceleration



$$\underbrace{\vec{v}}_{\text{vector}} = \vec{r}\,' = |\vec{r}\,'| \cdot \frac{\vec{r}\,'}{|\vec{r}\,'|} = \underbrace{|\vec{v}|}_{\text{magnitude}} \cdot \underbrace{\vec{T}}_{\text{direction}}$$

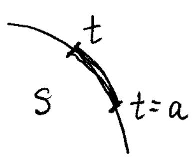
Recall that acceleration

$$\vec{a} = \vec{v}\,' = (|\vec{v}| \cdot \vec{T})'\,;$$

thus acceleration occurs because both the magnitude and the direction of the velocity change.

- Before we derive a formula for $\vec{a}$, let us write an equivalent expression for $|\vec{v}|$.

From Sec. 13.3 we have that



$$s(t) = \int_a^t |\vec{v}(\tilde{t})|\, d\tilde{t}$$

By the Fundamental Thm. of Calculus from Calc. I:

$$\frac{ds}{dt} = |\vec{v}(t)|. \qquad \left(\frac{d}{dx} \int_a^x f(u)\, du = f(x) \right)$$

Meaning:

Speed is just the rate at which traveled distance increases. So, we will use the relation $|\vec{v}| = s'$ below; thus

$$\vec{a} = (|\vec{v}| \cdot \vec{T})' = (s' \cdot \vec{T})' = s'' \cdot \vec{T} + s' \cdot \vec{T}\,'$$

Consider these terms one at a time.

1st term: $s'' \cdot \vec{T} = \underbrace{(s')'}_{} \cdot \vec{T}$

magnitude: rate of change of speed (i.e., how speed changes along the curve) | direction: along curve

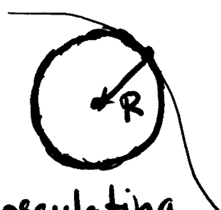
2nd term:

Sec. 13.3

$$s' \cdot \vec{T}' \equiv s' \cdot |\vec{T}'| \cdot \frac{\vec{T}'}{|\vec{T}'|} = s' \cdot \underbrace{\frac{|\vec{T}'|}{|\vec{r}'|}}_{} \cdot \underbrace{|\vec{r}'|}_{} \cdot \vec{N} =$$

(Sec. 13.3: curvature $\to \kappa(t)$ $\quad |\vec{v}| \equiv s'$)

$$= s' \cdot \kappa \cdot s' \cdot \vec{N} \equiv |\vec{v}|^2 \cdot \kappa \cdot \vec{N}.$$



osculating circle

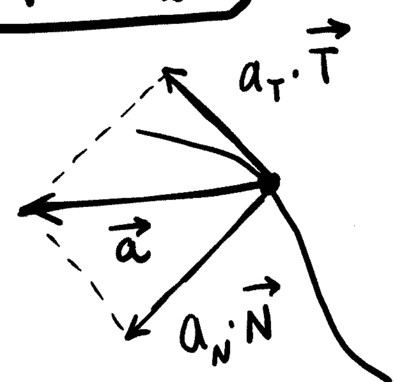
Interpretation: Since $\kappa = \frac{1}{R}$, this term is just the acceleration pointing towards the center of the osculating circle, just as in Example 1.

Combining both terms:

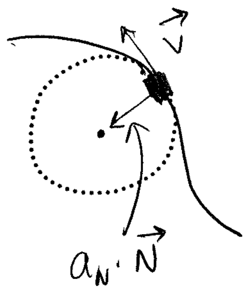
$$\boxed{\vec{a} = \underbrace{s''}_{\equiv a_T} \cdot \vec{T} + \underbrace{|\vec{v}|^2 \kappa}_{\equiv a_N} \cdot \vec{N} = a_T \cdot \vec{T} + a_N \cdot \vec{N}}$$

$\equiv a_T$ tangential component of acceleration = acceleration along the curve; due to change of speed

$\equiv a_N$ normal component of acceleration ($\perp$ to the curve) due to curve changing its direction

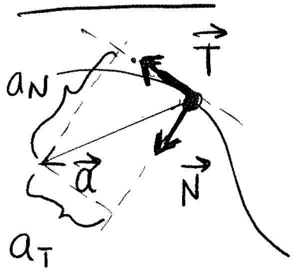


- Note 1: The normal part, $a_N \cdot \vec{N}$, of acceleration is also called centripetal acceleration. Its origin, again, is the same as the origin of the acceleration in Ex. 1 (uniform motion on a circle). It points towards the center of the osculating circle.



If you think of a car following a curved road, then the centripetal acceleration during a turn is provided by the force of friction between the car and the road. This is what makes the car stay on the road and not "fly away".

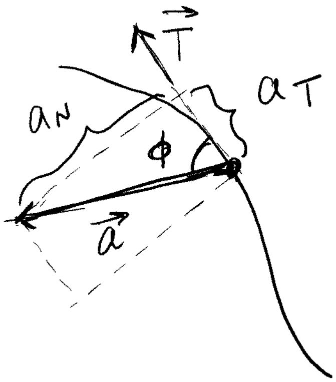
Note 2:



As a car moves on the road, at any moment the instantaneous unit vectors $\vec{T}$ and $\vec{N}$ form a "local coordinate system" for the car. These local coordinate vectors $(\vec{T}, \vec{N})$ are specific to the given road and a given point on it. In contrast, the unit vectors $(\vec{i}, \vec{j})$ are fixed and "know nothing" about the road. Depending on the situation, one can be convenient to view $\vec{a} = a_T \vec{T} + a_N \vec{N}$ or as $\vec{a} = a_x \vec{i} + a_y \vec{j}$.

Alternative expressions for a_T & a_N.

If one needs to compute $a_T = s''$ and $a_N = (s')^2 \kappa$, then there is an easier way than using these formulas.



From the picture, one has:

$$a_T = |\vec{a}| \cdot \cos\phi = |\vec{a}| \cdot \cancelto{1}{|\vec{T}|} \cdot \cos\phi$$

$$\underset{\text{Sec. 12.3}}{=} \vec{a} \bullet \vec{T} = \frac{\vec{a} \bullet \vec{r}\,'}{|\vec{r}\,'|} = \frac{\vec{r}\,'' \bullet \vec{r}\,'}{|\vec{r}\,'|}.$$

Similarly,

$$a_N = |\vec{a}| \cdot \sin\phi = |\vec{a}| \cdot \cancelto{1}{|\vec{T}|} \cdot \sin\phi$$

$$\underset{\text{Sec. 12.4}}{=} |\vec{a} \times \vec{T}| = \left| \vec{r}\,'' \times \frac{\vec{r}\,'}{|\vec{r}\,'|} \right| = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|}.$$

Thus:

$$\boxed{a_T = \frac{\vec{r}\,'' \bullet \vec{r}\,'}{|\vec{r}\,'|} \; ; \qquad a_N = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|}}$$

See Ex. 7 in the book for numbers.