Sec. 14.1 Functions of several variables

① Motivation and basic definitions

Most quantities in nature depend on more than 1 variable.
E.g., fuel efficiency of your car depends on such factors as:

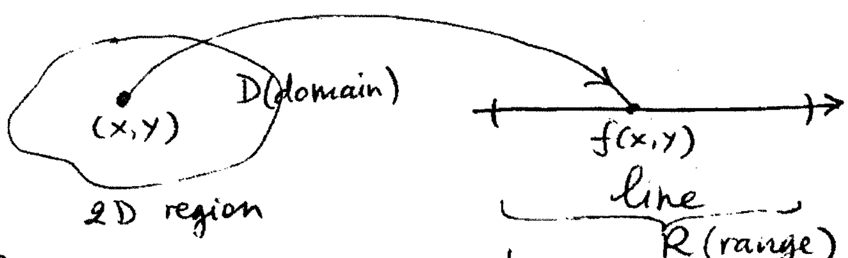
- tire pressure;
- weight load;
- velocity of wind relative to velocity of car;
- elevation angle of road.

So:

$$\underset{\text{fuel efficiency}}{E} = f(P, L, W, A, \ldots).$$

See also Exs. 2, 3 in book.

Def. of a function of 2 variables
(see book p. 855 for a rigorous form)



$f(x,y)$ is a rule that ~~for~~ to each point (x,y) in a 2D domain D puts in correspondence a point on a 1D line within some range R.

skip, refer to book

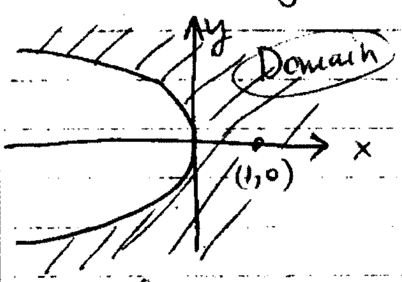
Ex. 1 (see also Ex. 1,4 in book)

Given $f(x,y) = x^2 + \sqrt{x+y^2}$, find:
1) $f(-2,3)$; 2) domain of f; 3) range of f.

Sol'n: 1) $f(-2,3) = (-2)^2 + \sqrt{-2+3^2} = 4 + \sqrt{7}$.

2) Domain of f is where each term in it is defined.
x^2 is defined for all x, y.
$\sqrt{x+y^2}$ is defined where $x+y^2 \geq 0$.

The boundary of the domain is:
$x+y^2 = 0$, $x = -y^2$.



We need $x+y^2 \geq 0$,
so pick any pt., say
$(x=1, y=0)$: $1+0^2 = 1 \geq 0$.

So $x+y^2 \geq 0$ is on that side of curve $x+y^2=0$ where $(1,0)$ is.

3) Range is what values f can take.
$f = \underbrace{x^2}_{\geq 0} + \underbrace{\sqrt{x+y^2}}_{\geq 0} \geq 0$.
So the range of f is $[0, \infty)$.

② Graphs of $f(x,y)$.

This is just the surface $z = f(x,y)$.

Ex. 1 (see also Ex. 5,6,8 in book) Sketch the graph of $f(x,y)$:

(a) $f(x,y) = x-2y+3$.

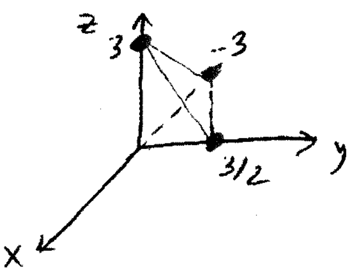
(b) $f(x,y) = 1-x^2$

(c) $f(x,y) = x^2+2y^2-3$

Sol'n: (a) $z = x-2y+3$

$x-2y-z+3=0$

This is a plane $\perp$ to $\langle 1,-2,-1\rangle$.



$z=0$: $x-2y+3=0$

$x=0 \Rightarrow -2y+3=0 \Rightarrow y=3/2$

$y=0 \Rightarrow x+3=0 \Rightarrow x=-3$

$x=y=0 \Rightarrow z=3$

This plane has the intercepts as above.

In general, $f(x,y)=ax+by+c$ for any a,b,c is the eq. of a plane. It is called a linear function of (x,y) (in analogy with $y=Ax+B$ being a linear function of x).

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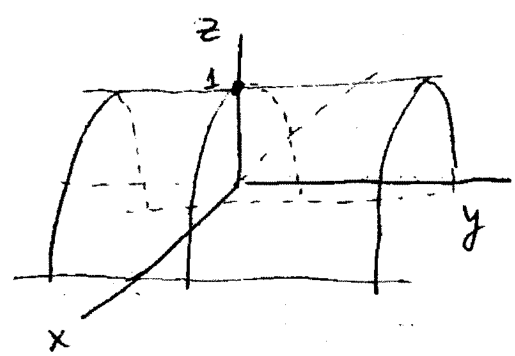
(b) $z = 1-x^2$

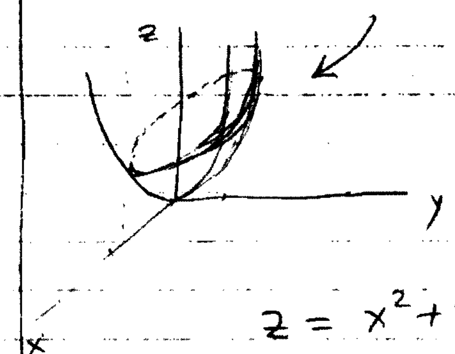
This is a parabolic cylinder with axis along y-axis (Sec. 12.6).



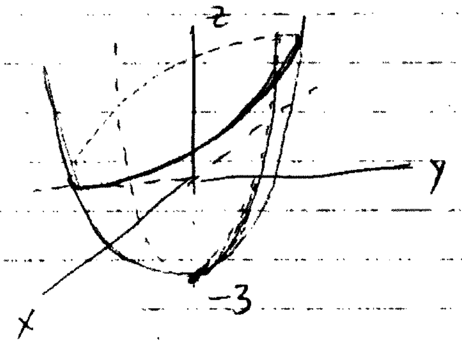
(e) $z = x^2 + 2y^2 - 3$

$z = x^2 + 2y^2$ is an elliptical paraboloid. (Sec. 12.6)

$$z = \frac{x^2}{1} + \frac{y^2}{(1/2)}$$



$z = x^2 + 2y^2 - 3$ is this paraboloid lowered by 3 units:



③ Level curves (contour plots)

If you cannot easily sketch $z = f(x,y)$, try to consider its traces in planes $z = K$ (= const). (Recall Lab 3.)

So, each trace is a curve:

$$z = K, \quad K = f(x,y).$$

The last equation is that of a 2D curve, which may be easier to sketch.

Ex. 2 Sketch:

(a) $(x+y)^2$ and

(b) $\sin(x+y)$

by considering their level curves.

(a) $(x+y)^2 = k.$

This can hold only for $k \geq 0$, so the surface $z=(x+y)^2$ lies on or above the xy-plane ($z \geq 0$).

$\underline{k=0}$: $(x+y)^2=0 \Rightarrow x+y=0 \Rightarrow y=-x$

$\underline{k=1}$ $(x+y)^2=1$

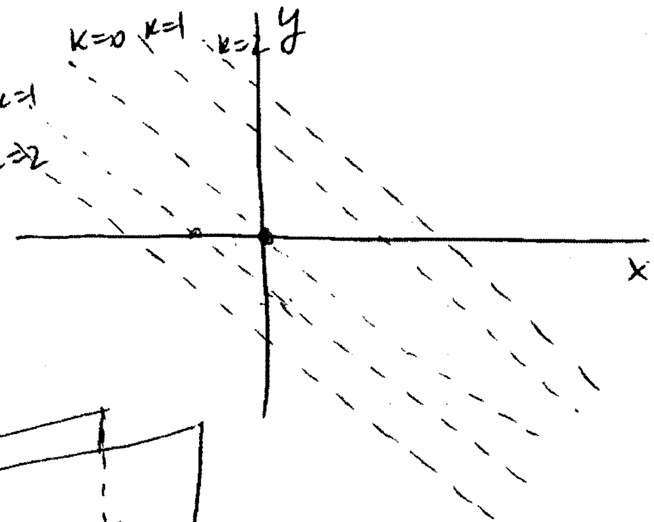
$x+y=\pm 1$

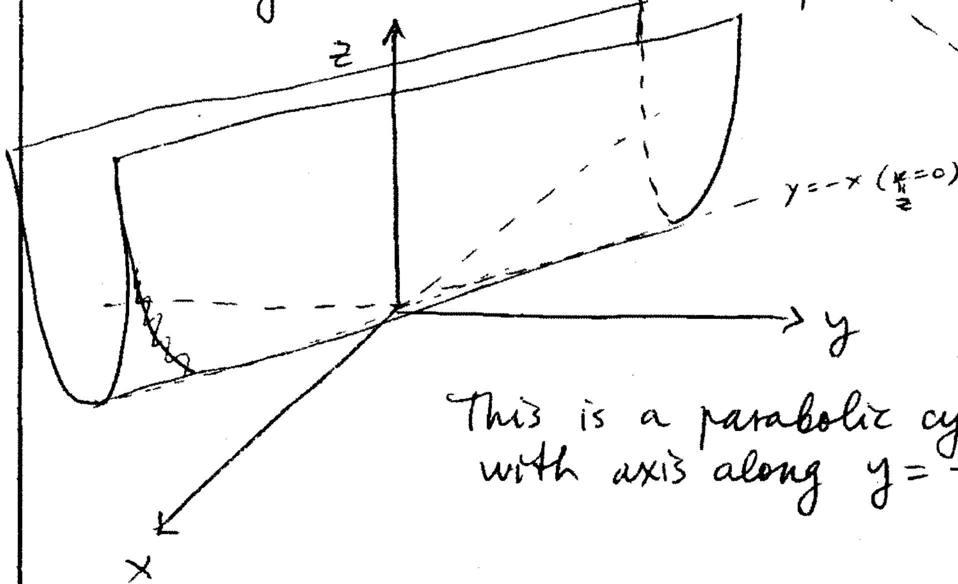
$y=-x \pm 1$

$\underline{k=2}$ $(x+y)^2=2$

$x+y=\pm\sqrt{2}$

$y=-x\pm\sqrt{2}$





This is a parabolic cylinder with axis along $y=-x$.

(b) $\sin(x+y) = k$

This can hold only for $-1 \leq k \leq 1$, so the surface $z=\sin(x+y)$ lies between planes $z=-1$ and $z=1$.

$k=0$ $\quad \sin(x+y)=0 \Rightarrow x+y = 0, \pm\pi, \pm 2\pi$, etc.

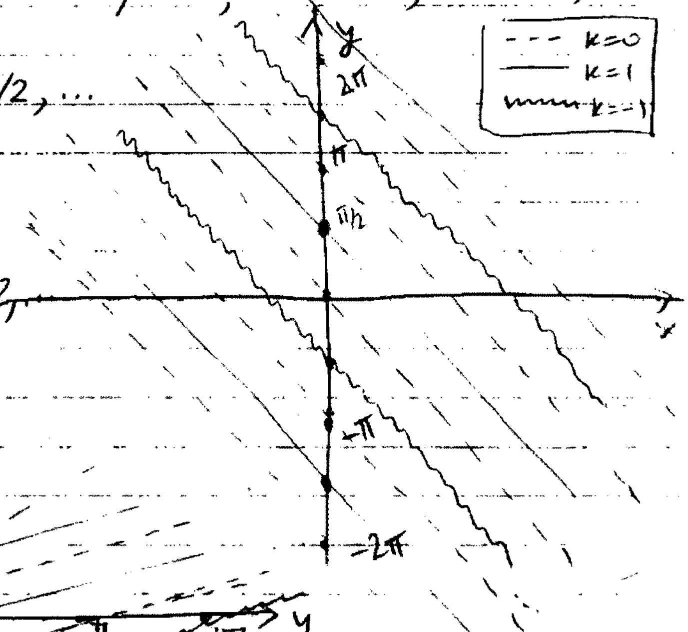
$\quad y=-x, \quad -x\pm\pi, \quad -x\pm 2\pi, \ldots$

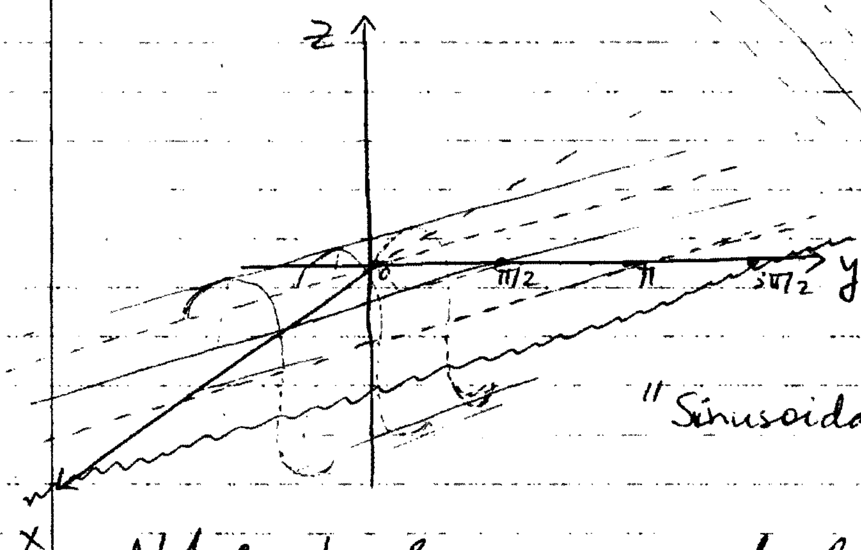
$k=+1$ $\quad \sin(x+y)=1$

$\quad x+y = \pi/2, \ 5\pi/2, \ldots$

$k=-1$ $\quad \sin(x+y)=-1$

$\quad x+y = -\pi/2, \ 3\pi/2, \ldots$





"Sinusoidal cylinder".

<u>Note:</u> Level curves cannot <u>always</u> be easily drawn. E.g.: $\sin x + \sin y = k$.
However, drawing level curves is usually a good first step to try.

<u>Notation</u>: A collection of level curves for various k (equidistant) is called a <u>contour plot</u> (or contour map).
A well-known example of a contour plot is a <u>topographic map</u>, where level curves denote regions of equal height.
Also: temperature or atmospheric pressure maps, etc.

Ex. 3 (a) Make a contour plot of a paraboloid $z = x^2 + y^2$.

(b) Based on this plot, comment how the steepness of this surface changes as the point on the surface moves away from (0,0,0).

Sol'n: (a) We need to draw several plots of $x^2+y^2 = k$. (obviously, $k \geq 0$).

$k=0$ $x^2+y^2=0 \to (0,0)$.

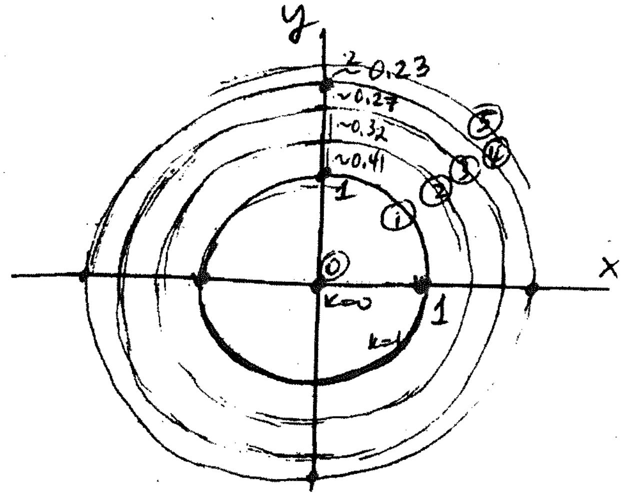
$k=1$ $x^2+y^2 = 1$
circle, radius $=1$

$k=2$ $x^2+y^2 = (\sqrt{2})^2$
radius $= \sqrt{2}$

$k=3$ $x^2+y^2 = 3 = (\sqrt{3})^2$
rad. $= \sqrt{3}$

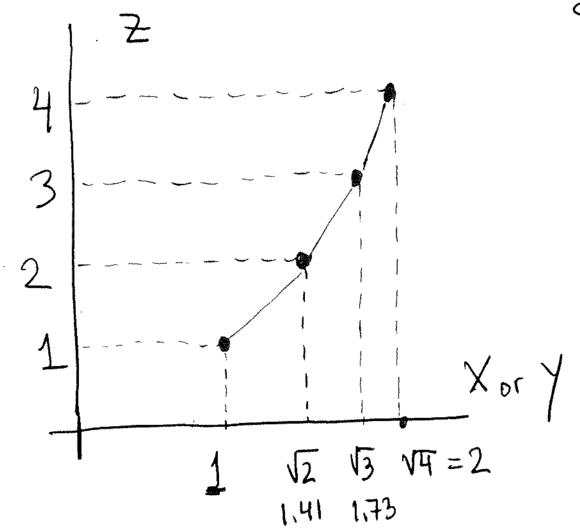
$k=4$ $x^2+y^2 = 4 = 2^2$
rad. $=2$

$k=5$ $x^2+y^2 = (\sqrt{5})^2$
rad $= \sqrt{5} \approx 2.23$



As $k \uparrow$, the circles become closer and closer.

(b) As we go from k-th level curve to the (k+1)th, we need to make smaller and smaller steps in the xy-plane, as $k \uparrow$.



$$\text{Slope} = \frac{\Delta z}{\Delta x}$$

Δz: = 1 by assumption

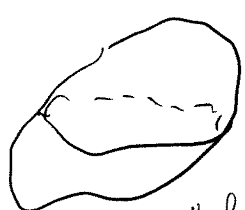
Δx: decreases

$\Rightarrow$ slope $\uparrow$ as $x \uparrow$ in this case.

Moral:

- **Level curves crammed <u>close</u> together indicate a <u>steep</u> surface;**
- **Level curves spaced <u>far apart</u> indicate a surface with <u>gentle slope</u>.**

(4) Functions of 3 variables



Examples:

"rho" → $\rho(x, y, z) \leftarrow$ density of a 3D object

$T(x, y, z) \leftarrow$ temperature of a 3D object

Level curves in 2D become <u>level surfaces</u> in 3D.

Example: $T(x, y, z) = k$ defines a surface of equal temperature inside a 3D object.

See also Ex. 15 in textbook.