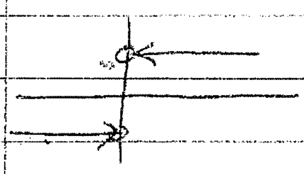
Sec. 14.2 Limits & continuity.

① Limits along curves.

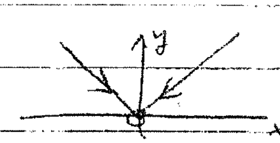
Before, we used to have the left and right limits:

$$\lim_{x \to +0} \frac{|x|}{x} = +1, \quad \lim_{x \to -0} \frac{|x|}{x} = -1$$

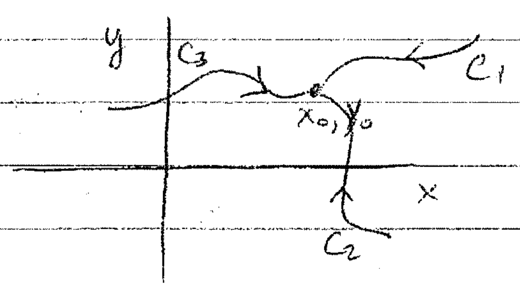


Or $\lim_{x \to +0} |x| = 0$

$\lim_{x \to -0} |x| = 0.$



For functions of 2 variables, instead of limits from left and right, we have limits along different curves:



If C is a curve defined by the parametric equations $x = x(t)$, $y = y(t)$, and t_0 is s.t. $x_0 = x(t_0)$, $y_0 = y(t_0)$, then

$$\lim_{\substack{(x,y) \to (x_0,y_0) \\ \text{along } C}} f(x,y) = \lim_{\substack{t \to t_0 \\ C:\, x = x(t),\, y = y(t)}} f(x(t), y(t)).$$

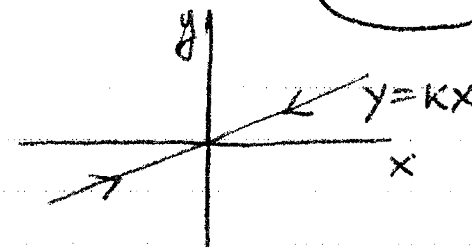
Ex. 1 Find the limits of $f(x,y) = \dfrac{xy}{x^2+y^2}$ along the lines

(a) $y = kx$, $k = \text{const}$

(b) $y = x^2$.

as $(x,y) \to (0,0)$.

(a) along $y = kx$



$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{xy}{x^2+y^2} =$$

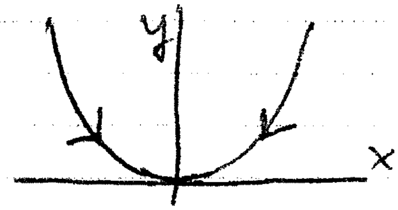
$$\lim_{x\to 0} \frac{x\cdot kx}{x^2+(kx)^2} = \lim_{x\to 0} \frac{k\cdot \cancel{x^2}}{\cancel{x^2}(1+k^2)} = \frac{k}{1+k^2}$$

(note that there is no y anymore!)

Note 1: The limit depends on the direction of the line along which we approach (0,0).

Note 2: We didn't need to use L'Hopital's rule to find $\lim_{x\to 0}$: simply cancelling out by x^2 was sufficient.

(b) along $y = x^2$



$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x^2}{x^2+(x^2)^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x^{\cancel{3}}}{\cancel{x^2}(1+x^2)}$$

$$= \lim_{x\to 0} \frac{x}{1+x^2} = \frac{0}{1+0} = 0.$$

Analog of Ex. 1(a) in 3D

Similarly, one can consider limit of $f(x,y,z)$ as $(x,y,z)\to(x_0,y_0,z_0)$ along some parametric curve:

$x = x(t)$, $y = y(t)$, $z = z(t)$.
In the HW you'll only need to consider <u>straight lines</u> approaching $(0,0,0)$:

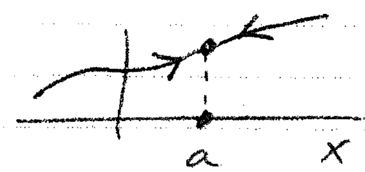
$\{x = at,\ y = bt,\ z = ct\}$ (see Sec. 12.5A).
Or if we redenote:

$$at = t_{new};$$
$$bt = \left(\tfrac{b}{a}\right) at = \left(\tfrac{b}{a}\right) t_{new} \equiv k \cdot t_{new}$$
$$ct = \left(\tfrac{c}{a}\right) at = \left(\tfrac{c}{a}\right) t_{new} \equiv l \cdot t_{new},$$

it suffices to use: $\{x = t,\ y = kt,\ z = lt\}$ for $k, l = \text{const}$.

(2) General limits of function of 2 variables

For $f(x)$, we said that $\lim_{x\to a} f(x)$ exists if $\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x)$:



For $f(x,y)$ we have a similar thm:

<u>Thm. 14.2.1</u> (a) If $f(x,y) \to L$ as $(x,y) \to (x_0,y_0)$ along any smooth curve, then we say that the general limit exists:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

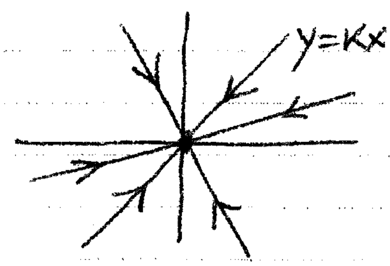
(b) If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ fails to exist along some (smooth) curve, or $\longrightarrow$

if $f(x,y)$ has <u>different limits</u> along <u>different curves</u> as $(x,y)\to(x_0,y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ <u>does not exist</u>.

about Ex. 1

> Thus, since in Ex. 1, $\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} f(x,y) = \dfrac{k}{1+k^2}$ <u>depended on k</u> (i.e. on the slope of the straight line), the <u>general</u> limit $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ D.N.E.

<u>Q</u>: Suppose that we can show that



$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} f(x,y) = L \quad \text{for } \underline{\text{all}}\ k,$$

i.e. the limit is the same along <u>any</u> straight lines $y=kx$.

Can we claim that the <u>general</u> limit,

$$\lim_{(x,y)\to(0,0)} f(x,y) = L \quad ?$$

<u>A</u>: No.

<u>Reason</u>: All straight lines $y=kx$ do <u>not</u> represent <u>all</u> smooth curves near $(0,0)$.

See the next example (but also the Note before it!).

Note: The situation described in this Example is RARE. It is shown FYI only.

Ex. 2 (a) Show that

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x^2y}{x^4+y^2} = 0 \quad \text{for all } k.$$

(b) Show that $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^2y}{x^4+y^2} = \frac{1}{2}$

(c) State whether the general limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \quad \text{exists.}$$

Sol'n: (a)

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2\cdot kx}{x^4+(kx)^2}$$

$$= \lim_{x\to0} \frac{k\cdot x^{\cancel{3}}}{\cancel{x^2}(x^2+k^2)} = \lim_{x\to0} \frac{kx}{k^2+x^2}$$

$$= \begin{cases} \underline{k\neq0}: & \dfrac{0}{k^2+0} = 0 \\ \underline{k=0}: & \displaystyle\lim_{x\to0} \frac{0}{x^2} = 0 \end{cases}$$

cannot plug in $x=0$,

exact 0; approaches 0

(b)

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2\cdot x^2}{x^4+(x^2)^2} =$$

$$= \lim_{x\to0} \frac{x^4}{x^4+x^4} = \lim_{x\to0} \frac{1}{2} = \frac{1}{2}.$$

(c) Since the limits along $y=kx$ and $y=x^2$ are different, then the general limit D.N.E.

Algorithm for checking if the general limit exists:

Find the limit along $y = kx$ (as in Ex. 1(a)).

- If it is different for different k, $\Rightarrow$ by Thm. 14.2.1 the general limit D.N.E.

- If it is the same for all k, you cannot conclude that the general limit exists, but need to do more tests, described in the next topic (topic 3).

Note 1 You do not need to test the limit along $y = x^2$ for the above algorithm.

Also, in HW, do not worry about $k=0$ vs $k \neq 0$.

That limit was considered in Ex. 1(b), 2(b) as an illustration only. Follow Ex. 1(a) instead. !

Note 2 You cannot stop if you get an expression 0/0 after plugging in $x=0, y=0$. The expression 0/0 simply indicates that you have to do more work, as described above.

Note 3 You cannot apply the L'Hopital's rule to $f(x,y)$ in the same way as you applied it to $f(x)$ in Calc. I. It is simply mathematically wrong.

(And, you didn't need to apply L'Hopital's rule in Ex. 1 & 2 above because simpler techniques sufficed.)

③ Other methods of finding limits of $f(x,y)$. (See the 2nd bullet of the ALGORITHM on p. 12-6)

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Note: They don't work always, but often they do.

Ex. 3 Using polar coordinates. ← must review before ~~Ex.~~ 4. (sec. 10.3)

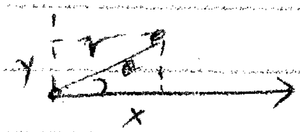
Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0.$

Sol'n: cartesian $\longrightarrow$ polar

$(x,y) \longrightarrow (r\cos\theta,\ r\sin\theta).$

Then $(x,y)\to(0,0)$ means:

$r\to 0,\ \theta =$ arbitrary.



$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta\cdot r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{r^{\cancel{4}\,2}\cos^2\theta\sin^2\theta}{\cancel{r^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}} = \lim_{r\to 0} \underbrace{r^2}_{\downarrow\, 0}\cdot\underbrace{(\cos^2\theta\sin^2\theta)}_{\text{any } \theta \text{ in } [0,1]}$$

$= 0$ (regardless of θ).

Ex. 4 Reducing problem to one variable.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{(x^4+y^4)}.$

Sol'n:

Using polar coords. is not convenient because $x^4+y^4 = r^4(\cos^4\theta+\sin^4\theta)$ depends on r and θ.

But notice that (x,y) enter this expression **only** in one combination:

$$(x^4+y^4).$$

So call $(x^4+y^4)=u$.

$(x,y)\to(0,0) \Rightarrow u\to 0$.

Thus

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{x^4+y^4} = \lim_{u\to 0} \frac{\sin u}{u} = 1 \text{ (Sec. 3.3).}$$

Ex. 5 Algebraic manipulation

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x-y)}{\cancel{(x+y)}} =$$

$$= \lim_{(x,y)\to(0,0)} (x-y) = 0-0 = 0$$

(since there seems to be no reason for $(x-y)$ not to have a limit at $(0,0)$).

Final note: In this Section we focused on differences between 1D & 2D limits. We also stressed examples where limits depend on the curve along which the limit is found. These examples are important both conceptually and in some applications (even one later in this course!). However, "most" functions of 2 variables tend to be continuous (= "general limit exists"). Examples: polynomials, sin, cos, $e^{\cdots}$.