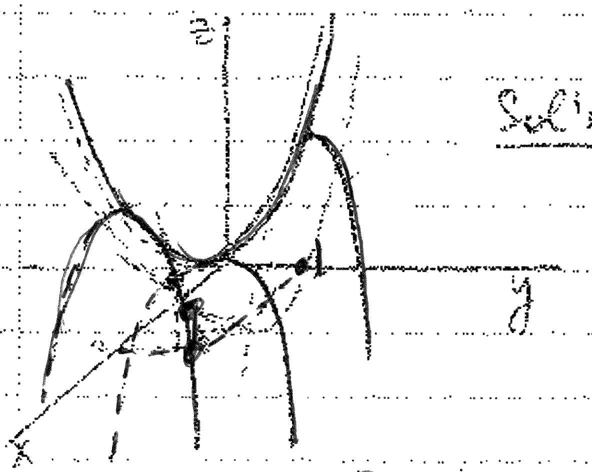
Sec. 14.3, Partial derivatives

These generalize derivatives of a function of one variable.

① Slopes of a surface along x- and y-directions.

Ex. 1 Find the slopes of $f(x,y) = x^2 - y^2$ at point $(2,1)$ in the directions of x- and y-axes.



Q: What surface is this?

Sol'n:

1) To find the slope along x-axis @ $(x=2, y=1)$, we cut the surface with plane

$$y = 1$$

In the cross-section, we obtain a trace: $z = x^2 - 1^2 = x^2 - 1$.

We now find the slope of this parabola at $x=2$ (still, of course, along x-axis):

$$\left.\frac{dz}{dx}\right|_{y=1} = 2x, \quad \Rightarrow$$

$$\left.\frac{dz}{dx}\right|_{\substack{y=1\\x=2}} = 2 \cdot 2 = 4. \leftarrow \text{Slope @ } (2,1) \text{ along x-direction.}$$

Notation:

$$\boxed{\frac{d f(x, y=y_0)}{dx} \stackrel{\text{def.}}{=} \frac{\partial f(x, y_0)}{\partial x} = f_x(x, y_0)}, \quad \text{or}$$

Likewise, $\left.\dfrac{dz(x,y_0)}{dx}\right|_{y=y_0} = \dfrac{\partial z(x,y_0)}{\partial x} = z_x(x,y_0)$.

the slope of $f(x,y)$ along the x-axis

Name: $f_x(x_0,y_0)$ ← partial derivative of f w.r.t. x

(implied: when y is kept constant)

In our example:

$$\frac{\partial f(x,1)}{\partial x} = 2x, \quad \frac{\partial f(2,1)}{\partial x} = 4.$$

See also Figs. 2 & 4 in textbook.

2) Similarly, the trace in the plane $x=2$ is

$$z = 2^2 - y^2 = 4 - y^2.$$

The slope in the y-direction is:

$$\left.\frac{dz}{dy}\right|_{x=2} = (4-y^2)' = -2y$$

$$\left.\frac{dz}{dy}\right|_{\substack{x=2\\y=1}} = -2\cdot 1 = -2.$$

$\frac{\partial f}{\partial y}$ is the slope of $z = f(x,y)$ along the y-axis

Using the notations for partial derivative:

$$\frac{\partial f(2,y)}{\partial y} = -2y; \quad \frac{\partial f(2,1)}{\partial y} = -2.$$

See Figs. 3 and 5 in textbook.

(PD)

Def: Partial derivative of f w.r.t. x @ (x_0,y_0):

$$f_x(x_0,y_0) \equiv \frac{\partial f(x_0,y_0)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

Partial derivative of f w.r.t. y @ (x_0,y_0):

$$f_y(x_0,y_0) \equiv \frac{\partial f(x_0,y_0)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

Mnemonic rule:

- To find f_x, regard y as a constant and differentiate $f(x,y)$ w.r.t. x.

• Similarly for f_y.

Ex. 2   Find f_x, f_y, and evaluate them at $(x_0, y_0) = (2, 1)$ for:

(a) $f(x,y) = x^3 + x^2 y^4 - 7y^5$

(b) $f(x,y) = x \exp[y/x] \equiv x \cdot e^{(y/x)}$

Sol'n:

(a) $f_x = 3x^2 + 2x \cdot y^4 - 0 = 3x^2 + 2xy^4$

$f_y = 0 + x^2 \cdot 4y^3 - 7 \cdot 5 \cdot y^4 = 4x^2 y^3 - 35 y^4.$

$f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^4 = 16$

$f_y(2,1) = 4 \cdot 2^2 \cdot 1^3 - 35 \cdot 1^4 = -19$

(b) $f_x = (x)_x \cdot \exp(y/x) + x \cdot (\exp(y/x))_x$

$= 1 \cdot \exp(y/x) + x \cdot (-\frac{y}{x^2}) \exp(y/x) =$

$= (1 - \frac{y}{x}) \cdot \exp(y/x).$

$f_y = x \cdot (\exp(y/x))_y = x \cdot \frac{1}{x} \cdot \exp \frac{y}{x} = \exp(y/x).$

$f_x(2,1) = (1 - \frac{1}{2}) e^{1/2} = \frac{1}{2} e^{1/2}$

$f_y(2,1) = e^{1/2}.$

(2) Some comments

Note 1   As the ordinary derivative is a rate of change, so are partial derivatives:

$f_x(x_0, y_0)$ is rate of change of f @ (x_0, y_0) along x-axis;

$f_y(x_0, y_0)$ ———————————— y-axis.

(Earlier we said that part. derivatives are slopes. But a slope and a rate of change are synonymous!)

Note 2. $f(x,y)$ can be defined by a table of values (see Sec. 14.1). Then f_x and f_y can be found as rates of changes along rows and columns of that table.

! **MUST READ** the Example at the beginning of Sec. 14.3 about finding part. derivatives from a table.

Note 3. Partial derivatives of functions of 3 and more variables are defined and computed similarly. E.g., to find $f_x(x,y,z)$, regard y and z as constants and differentiate w.r.t. x.
See Ex. 5 in book.

Note 4. Similarly to how the ordinary derivative $\frac{dy}{dx}$ can be found implicitly from an equation $F(x,y)=0$ (Sec. 3.5), partial derivatives z_x and z_y can also be computed implicitly from an equation $F(x,y,z)=0$.

E.g.: $x^2+y^2+z^2=1$ defines a sphere.
So think of z as $z(x,y)$. Then take $\partial/\partial x$:

$$\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(y^2)+\frac{\partial}{\partial x}(z^2)=\frac{\partial}{\partial x}1 \Rightarrow$$

$$2x+0+2z\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=-\frac{x}{z}.$$

⇛ MUST ALSO SEE Ex. 5 in book.
We will revisit this in Sec. 14.5, where I'll expect you to have read that example.

③ Higher-order partial derivatives. (PD)

For $f(x)$: $\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$ ← Second derivative

For $f(x,y)$: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv (f_x)_x \equiv f_{xx}$

Similarly: $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv (f_y)_y \equiv f_{yy}$

Second-order PD:

Mixed partial derivatives
$$\begin{cases} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx} \\ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy} \end{cases}$$

Note the order in these different notations!

∂-notations and subscript notations have reversed order of x,y.

Third-order PDs can be defined similarly:

e.g.,
$$f_{xyy} = (f_x)_{yy} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right).$$

Ex. 3 Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ for
$$f = x^3 + x^2y^4 - 7y^5$$

Sol'n: From Ex. 2: $f_x = 3x^2 + 2xy^4$
$f_y = 4x^2y^3 - 35y^4$

$f_{xx} = 3\cdot 2x + 2y^4$

$f_{xy} = (f_x)_y = 2x\cdot 4y^3 = \boxed{8xy^3}$

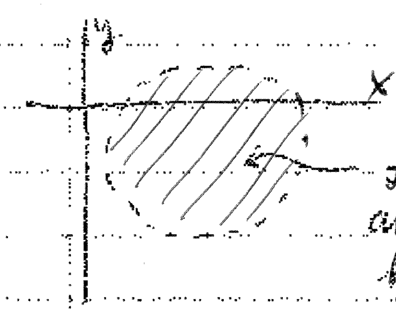
$f_{yx} = (f_y)_x = 4\cdot 2x\cdot y^3 = \boxed{8xy^3}$

$f_{yy} = 4x^2\cdot 3y^2 - 35\cdot 4y^3$

Note: $f_{xy} = f_{yx}$!

This is not a coincidence but a general fact:

Clairaut's Theorem:



f_{xy}, f_{yx} are continuous here

Let f_{xy} and f_{yx} exist and be continuous in some open region of the xy-plane. Then

$$f_{xy} = f_{yx} \text{ in that region.}$$

An example where f_{xy}, f_{yx} are discontinuous and hence not equal, will be considered in Lab 4. In such situations, the Mnemonic Rule from the bottom of p. 13-2 does not work, and one will need to use the Definitions of part. derivatives from that page.

(4) Partial differential equations

In Calc. I you must have seen an ORDINARY differential equation for the amount of money in a bank account:

$$\underset{\text{"ordinary derivative"}}{\frac{dM}{dt}} = r \cdot M$$

M ← amount

r ← interest rate

There are many examples of ordinary diff. eqs.

E.g.,

$$\underbrace{\frac{dN}{dt}}_{\text{Rate of change of number of bacteria}} = F(N, t)$$

F: some function

N: number of bacteria in a dish

t: time

Knowing the rate of change of N and the value $N(t=0)$ allows us to find $N(t)$ at any later time.
This is similar to the last example in Sec. 13.2, but more complicated (take Math 230 or Math 271).

Another example:

Newton's Cooling Law $\rightarrow$ $$\frac{dT}{dt} = -\text{const} \cdot (T - T_{ambient})$$

T: temperature of a small cup of coffee

But what if the cup of coffee is not small, i.e. T depends on time t and location (x, y, z) inside cup?
One can show that the Newton Cooling Law becomes:

$$\boxed{\frac{\partial T}{\partial t} = \begin{pmatrix}\text{another}\\ \text{const}\end{pmatrix} \cdot \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)}$$ **Heat Equation**

Partial diff. equations (involve partial derivatives).

1D Heat Equation: $$\frac{\partial T}{\partial t} = \text{const} \cdot \frac{\partial^2 T}{\partial x^2}$$

Another famous partial diff. eq.:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ **1D Wave Equation.**

c = wave speed