Sec. 14.5: Chain Rule

1 Explicit differentiation

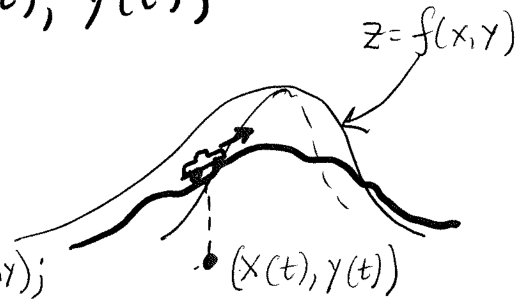
Recall from Calc. I:

If $z = f(x)$ and $x = x(t)$, then

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \quad \leftarrow \text{Calc. I Chain Rule}$$

Our goal in this topic will be to **extend** this 1-variable rule (i.e., each of f and x depend only one 1 variable) **to the multi-variable case.**

Case 1(a): $z = f(x(t), y(t))$



Interpretation:

Car on the road;

Road is on the hill $z = f(x,y)$;

In (x,y)-plane, the car's position is given by $(x(t), y(t))$.

One may then ask: ***How does the elevation (i.e. z-coordinate) of the car changes with t?***

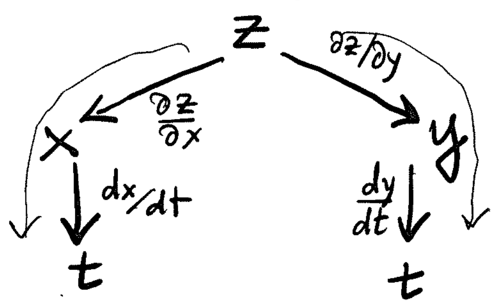
Formula:

$$\frac{dz}{dt} \equiv \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

"same as"

See Ex. 1/book for numbers.

Tree diagram:



Comments:

1) Always write the Tree Diagram before writing out the formula.
In the Tree Diagram, trace all paths that lead from z to t (2 such paths in the case above).

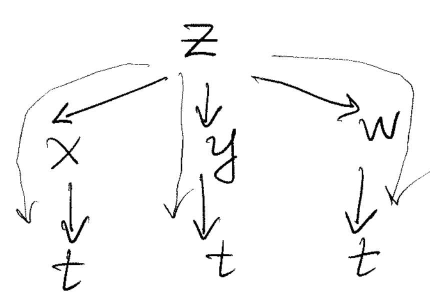
2) Note: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ are partial derivatives, but $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ are ordinary derivatives. Why?
Because $f(x,y)$ depends on more than one variable, $\Rightarrow$ need partials.
On the other hand, each of $x(t), y(t)$ depends on one variable, $\Rightarrow$ can only have ordinary derivative.

3) If f depends one more than 2 variables, the formula is similar:

$$\boxed{\frac{d f(x(t), y(t), w(t))}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dt}}$$

etc.

Tree Diagram:



(3 paths from z to t)

Case 1(b): In the last formula on the previous page, let's set $\boxed{w(t) = t}$. Then we have:

$$\frac{d\, f(x(t), y(t), t)}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial w}\cdot \boxed{\frac{d\cancel{w}}{dt}}$$

but: $\boxed{w=t}$ $\quad \frac{dt}{dt} = 1$

Thus:

$$\boxed{\frac{d\, f(x(t), y(t), t)}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial t}}$$

Ex. 1 $\quad f(x,y,t) = tx^2 + \sin(xy)$, $x = 3\sqrt{t}$, $y = e^{5t}$.

Find df/dt.

Sol'n: Use the formula of Case 1(b):

$$\frac{df}{dt} = \Big(\underbrace{t\cdot 2x + y\cdot\cos(xy)}_{\partial f/\partial x}\Big)\cdot\underbrace{\frac{3}{2\sqrt{t}}}_{dx/dt} +$$

$$+ \underbrace{x\cdot\cos(xy)}_{\partial f/\partial y}\cdot\underbrace{5e^{5t}}_{dy/dt} + \underbrace{x^2}_{\partial f/\partial t}$$

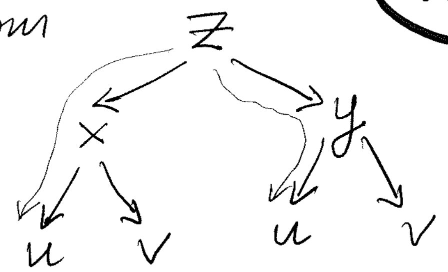
(Can simplify, but this won't be required on quiz/test.)

Case 2

$z = f(x,y)$, $\quad x = x(u,v)$
$\quad\quad\quad\quad\quad\quad y = y(u,v)$

To obtain the formula for, say, $\frac{\partial z}{\partial u}$, draw the Tree Diagram.

There are 2 paths leading from z to u. So:



$$\boxed{\frac{\partial f(x(u,v),\, y(u,v))}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

and similarly for $\partial f/\partial v$.

<u>Note:</u> All derivatives here <u>are partial</u> (not ordinary) because all variables: f, x, y depend on more than 1 variable (see the Tree Diagram).

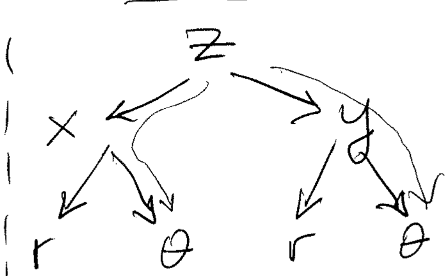
<u>Ex. 2</u> (similar to Ex. 3 in book):

Let $z = x^5 e^{y^2}$, $x = r\cos\theta$, $y = r\cdot\sin\theta$.

Find $\partial z/\partial\theta$.

<u>Sol'n:</u>

<u>Tree:</u>



$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

from Tree

$$= \left(5x^4\cdot e^{y^2}\right)\cdot\left(-r\cdot\sin\theta\right) + \left(x^5\cdot 2y\cdot e^{y^2}\right)\cdot\left(r\cdot\cos\theta\right).$$

One can, of course, simplify this expression, but this will not be required on test/quiz.

Case 2 for for functions of 3 or more variables is handled similarly; See Ex. 4,5/book.

2 Implicit differentiation.

Review of Calc. I situation (recall that you were required to read Ex. 5 in Sec. 14.3 to review this material):

Find the slope of circle $x^2+y^2=R^2$ at a given point (x,y) on the circle.

Sol'n: On the circle, we have $y=y(x)$. Differentiate the equation of circle w.r.t. x:

$$\frac{d}{dx}\left(x^2+(y(x))^2=R^2\right) \Rightarrow$$

$$2x+2y\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}.$$

See also Ex. 8 in textbook (Sec. 14.5).

We can generalize this from a circle to any curve:

$\underbrace{f(x,y)=0}_{}$ defines some curve $y=y(x)$

($x^2+y^2-R^2=0$ in the above case of the circle).

$$\frac{d}{dx}\left(f(x,y(x))=0\right) \Rightarrow \frac{\partial f}{\partial x}\cdot\overset{1}{\cancel{\frac{dx}{dx}}}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

Case 1 in topic 1

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x & & y \arrow[d] \\ & & x \end{tikzcd}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}.$$

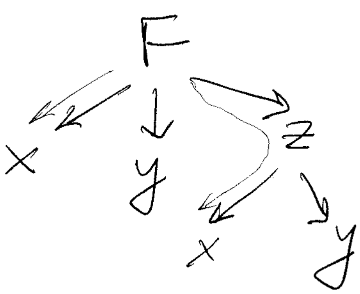
And now we can generalize to 3 variables:

$\underbrace{F(x, y, z) = 0}$ defines some surface $z = z(x,y)$.

(think of sphere $x^2+y^2+z^2-R^2=0$)

To find the slope of this surface along, say, x-axis:

$$\frac{\partial}{\partial x}\Big(F(x, y, z(x,y)) = 0\Big) \Rightarrow$$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}.$$

See posted page 13-4 and Ex. 9 in book.

(3) Application to rates of change

MUST READ Ex. 2 in textbook.

(4) Application to partial differential equations

Ex. 3 Show that for any function $f(u)$, the function $z = f(x+2y)$ satisfies a partial diff. equation $2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$.

Interpretation (vague): This P.D. equation is a special case of the Wave Eqn. (sec. 14.3). The problem "shows" (vague!) that a disturbance of any shape can propagate along a tight string (vague!!)

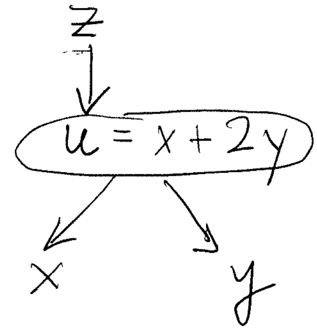
Sol'n:

$$\frac{\partial z}{\partial x} = \frac{d\overset{f(u)}{z}}{du}\cdot\frac{\partial u}{\partial x} = \frac{df}{du}\cdot\frac{\partial \overset{(x+2y)}{u}}{\partial x}$$

$$= \frac{df}{du}\cdot 1$$

$$\frac{\partial z}{\partial y} = \frac{df}{du}\cdot\frac{\partial \overset{(x+2y)}{u}}{\partial y} = \frac{df}{du}\cdot 2 .$$

similarly



Thus,

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2\cdot\frac{df}{du}\cdot 1 - \frac{df}{du}\cdot 2 = 0. \quad \checkmark$$

See also Ex. 6 / book (it is more complicated).