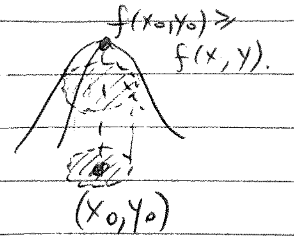
Sec. 14.7. Max and min values

① ⟵ local maxima & minima.

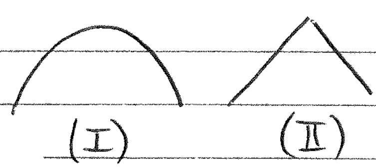
Def: (similar to that for $f(x)$).



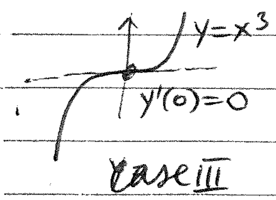
$f(x,y)$ has a local max at (x_0,y_0) if $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in some disk centered at (x_0,y_0).
(similar for loc. min.)
See Ex-1 in book.

Q: How do we find local extrema (i.e. max and minima) of $f(x,y)$?

Recall Calc. I: If $f(x)$ has a local extremum at $x=x_0$, then either $f'(x_0)=0$ (case I) or $f'(x_0)$ doesn't exist (case II)



But note: $f'(x_0)=0$ does not always mean that x_0 is an extremum:



Notation: All points where $f'(x)=0$ or $f'(x)$ doesn't exist are called critical points. Not all critical points are extrema, but any extremum is a crit. point.

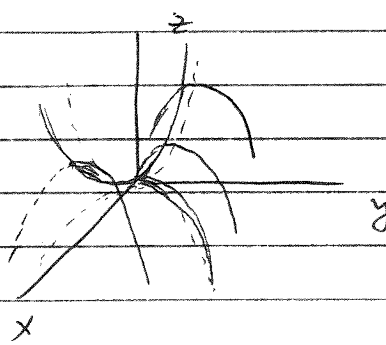
For $f(x,y)$: If $f(x,y)$ has a local extremum at (x_0,y_0), then either $\vec{\nabla} f(x_0,y_0)=\vec{0}$ (i.e. $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$) or $\vec{\nabla} f(x_0,y_0)$ doesn't exist.
Points where $\vec{\nabla} f=\vec{0}$ or $\vec{\nabla} f$ doesn't exist are called critical points of $f(x,y)$.

As with $f(x)$, $f(x,y)$ does not necessarily have an extremum at every critical point.

E.g., $f = x^3 + y^3$ has $\vec{\nabla} f = \vec{0}$ at $(0,0)$ but has no max. or min there (similarly to $f(x) = x^3$ at 0 in 1D).

But in 2D, we can have a possibility that does not occur in 1D: we can have a critical point that is a saddle point.

$z = x^2 - y^2$



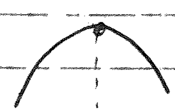
- $\vec{\nabla} f = \vec{0}$ at $(0,0)$
- $f(x,0)$ has a loc. min
- $f(0,y)$ has a loc. max

$(0,0)$ is neither a loc. max nor a loc. min. — it is a saddle point.

In general, $z = f(x,y)$ has a saddle at (x_0, y_0) if there are two directions, such that f has a local max. along one and a local min along the other at (x_0, y_0).

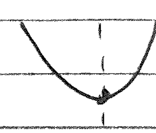
Q: How can we determine analytically whether a crit. point is a min, max, saddle, or none of the above?

For $f(x)$, we had a 2nd-derivative test:



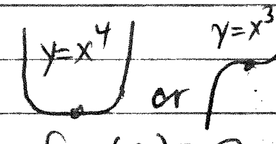
$f_{xx}(x_0) < 0$

loc. max



$f_{xx}(x_0) > 0$

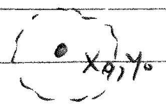
loc. min



$f_{xx}(x_0) = 0$

no information.

For $f(x,y)$ ← 2nd-derivatives test.
Let (x_0, y_0) be a critical point where $f_x = f_y = 0$
(i.e., we exclude the case "$\vec{\nabla} f$ doesn't exist")
and where f_{xx}, f_{xy}, f_{yy} exist and are continuous
in a disk containing (x_0, y_0).



Let
$$D = \begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}$$
$$\equiv \left(f_{xx} \cdot f_{yy} - f_{xy}^2 \right)\Big|_{@(x_0,y_0)}$$

Then:
(a) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, $\Rightarrow$ (x_0,y_0) is a local min
(b) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, $\Rightarrow$ (x_0,y_0) is a loc. max.
(c) If $D < 0$, then (x_0,y_0) is a saddle point.

If $D = 0$, this test yields no information about the type of the critical point.

Ex. 1. Find the critical points of $z = x^3 - 12xy + 8y^3$.
Determine whether the function has a local min, max, or a saddle at each critical point.
(See also Ex. 3 in book)

Sol'n: 1) Find critical points:
$\vec{\nabla} f = \vec{0}$ or $\vec{\nabla} f$ doesn't exist.
Since f is a polynomial, $\vec{\nabla} f$ exists always, so we need to find where $\vec{\nabla} f = \vec{0}$, or

"and" $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \Rightarrow$

$\begin{cases} x^2 - 4y = 0 \\ x - 2y^2 = 0 \end{cases} \Rightarrow \begin{cases} (2y^2)^2 - 4y = 0 \\ x = 2y^2 \end{cases} \Rightarrow$

$$\begin{cases} y^4 - y = 0 \\ x = 2y^2 \end{cases} \Rightarrow \begin{cases} y(y^3-1) = 0 \\ x = 2y^2 \end{cases} \Rightarrow \begin{cases} y=0 \ \underline{\underline{or}}\ y=1 \\ x = 2y^2 \end{cases}$$

$$\Rightarrow (y=0,\ x=2\cdot 0) \text{ or } (y=1,\ x=2\cdot 1^2).$$

Thus, $(0,0)$ and $(2,1)$ are the critical points.

2) Use the 2nd-der. test to find their type.

$$D = \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix} = 288xy - 144.$$

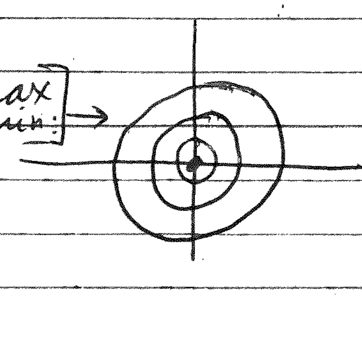
$$\begin{array}{ll} @(0,0) & D = 288\cdot 0 - 144 < 0 \Rightarrow \text{saddle.} \\ @(2,1) & \left.\begin{array}{l} D = 288\cdot 2\cdot 1 - 144 > 0 \\ f_{xx}(2,1) = 6\cdot 2 > 0 \end{array}\right\} \Rightarrow \text{loc. min.} \end{array}$$

<u>Thus</u>: $(0,0)$ is a saddle.
$(2,1)$ is a loc. min.

<u>Q</u>: If $D=0$ (the 2nd-der. test yields no info), can we still determine the type of the critical point?
<u>A</u>: Yes, from a contour plot.

We just need to ~~see~~ learn what the contour plot looks like near: (a) a loc. max or min, (b) a saddle, (c) a crit. point that is not an extremum.

(a) Loc. min: $z = x^2 + y^2$.
Contour plot of ~~a loc max~~ or min: $\rightarrow$



(will look the same for a loc. max: $z = -(x^2+y^2)$.)

(b) Saddle $\quad z = x^2 - y^2$

$z = k$: $\quad k = 0 \quad x^2 - y^2 = 0$

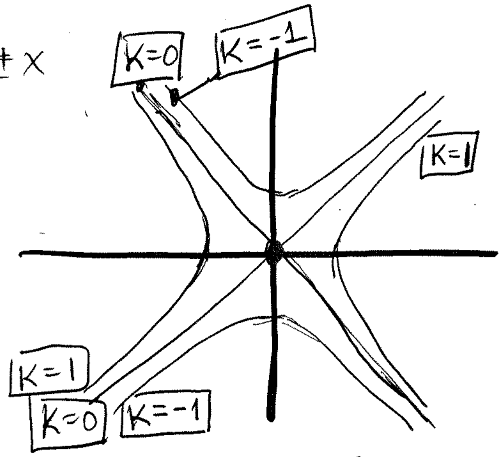
$y^2 = x^2 \Rightarrow y = \pm x$

$k = 1 \quad x^2 - y^2 = 1$

hyperbola type I
(Lab 1 + sec. 12.6)

$k = -1 \quad x^2 - y^2 = -1$

hyperbola type II



Contour plot of a saddle

(c) Not an extremum $\quad z = x^3 + y^3$

$z = k$: $\quad k = 0 \quad x^3 + y^3 = 0$

$y^3 = -x^3 \Rightarrow y = -x$

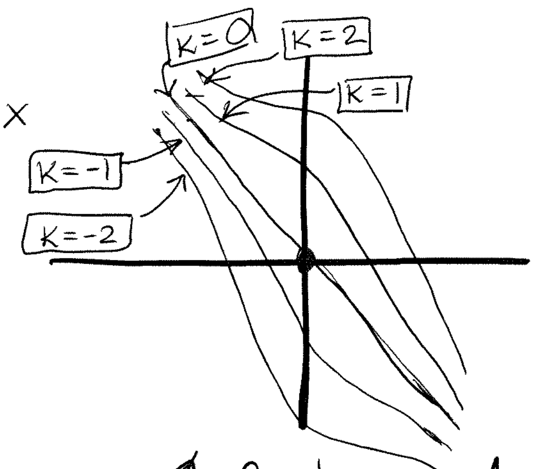
$k = 1 \quad x^3 + y^3 = 1$

$y^3 = -(x^3 - 1) \Rightarrow$

$y = -\sqrt[3]{x^3 - 1}$

$k = -1 \quad x^3 + y^3 = -1 \Rightarrow$

$y = -\sqrt[3]{x^3 + 1}$ etc.



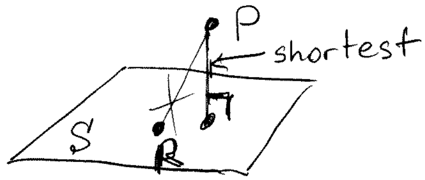
Contour plot of a cr. point that is not an extremum

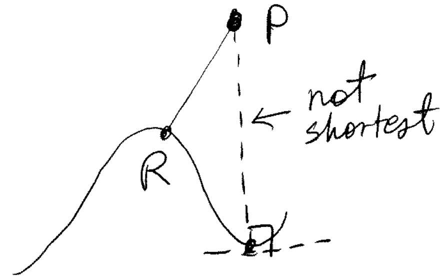
Ex. 2 Distance from a point to a **surface** (see also Ex. 5/book)

This Example will be referenced in Lab 6.

In Sec. 12.5B we had a formula for the distance between a point and a plane.

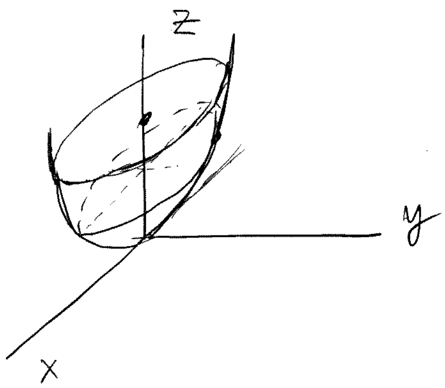
Now we will generalize "the plane" to an arbitrary (curved) surface.





For the plane, the distance is measured along the line $\perp$ to the plane. This distance is <u>the shortest</u> among all distance $|PR|$ for any pt. R on the plane.

For an arbitrary surface, we also declare the distance from a pt. to the surface as <u>the shortest</u> among all distances $|PR|$.



Set up the equations to find the distance from $P=(11,12,13)$ to the paraboloid $z=x^2+2y^2$.

1) distance to <u>any</u> point (x,y,z):

$$d=\sqrt{(x-11)^2+(y-12)^2+(z-13)^2}$$

2) account for the fact that the point is on the paraboloid $z=x^2+2y^2$ (i.e., <u>replace</u> z with (x^2+2y^2)):

$$d(x,y)=\sqrt{(x-11)^2+(y-12)^2+(x^2+2y^2-13)^2}$$

To find its minimum, we set up the equations as in Ex. 1 : $\to \begin{cases} \frac{\partial}{\partial x} d(x,y) = 0 \\ \frac{\partial}{\partial y} d(x,y) = 0 \end{cases}$ and proceed as in that example to find all critical pts. If there is more than 1 cr. pt, we find the minimum value of $d(x_c, y_c)$ among these values for all cr. pts.

This brings us to the next topic.

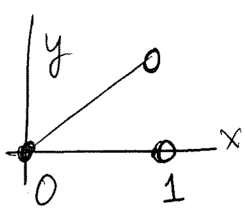
② Absolute maxima and minima

Let's recall from Calc. I (Sec. 4.1):

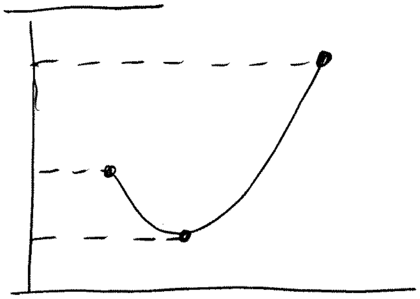
Extreme Value Thm for $y=f(x)$:

If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is guaranteed to reach its max. and min. values there.

Note 1: We want the boundaries a and b included; otherwise, we are not guaranteed to reach an extremum (e.g., $y=x$ on $[0,1)$).



Note 2: The abs. max (or abs. min) does not have to occur at a local max (or local min, respectively)! It may occur at the boundary.



Algorithm for finding abs. extrema for $y = f(x)$ (Calc. I)

1) Find all cr. pts $c_1, c_2, \ldots$ on $[a, b]$.
2) Select the min and max values from the finite set $\{f(c_1), f(c_2), \ldots, f(a), f(b)\}$.

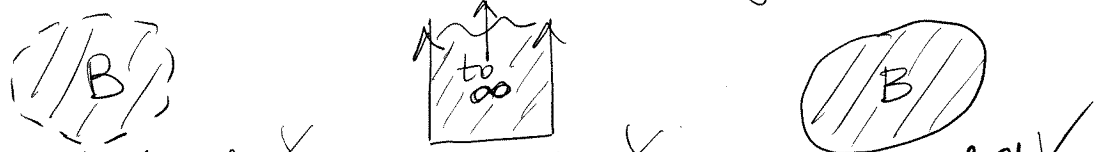
For $z = f(x,y)$, the process of finding abs. extrema is conceptually similar, but more complicated technically.

Extreme Value Thm. for $z = f(x,y)$

If $f(x,y)$ is continuous on a close & bounded region B in the xy-plane, then $f(x,y)$ is guaranteed to reach it min and max values in B.

Q: What is close & bounded ?

A: All of the boundary is included (=closed), and no part of B is at infinity.



not closed, Bounded ✗ | closed, not bounded ✗ | closed & bounded ✓

Algorithm for finding abs. extrema of $z = f(x,y)$ (Calc. III)

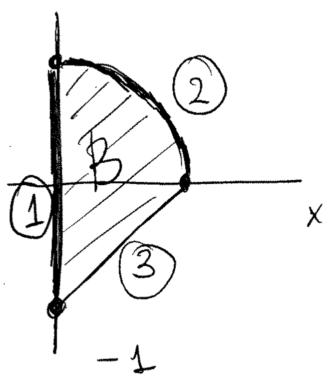
1. Find all cr. pts in B: $c_1, c_2, \ldots$
2. Find the abs. min. & max, on the boundary of B (this is a 1D problem that we need to solve with the Calc. I Alg.)
3. Find the min & max values among those found in 1. and 2.

Note 1 Step ② of the Calc. III Algorithm is new compared to the Calc. I Algorithm. It is also the most time consuming to do.

Note 2: To find absolute extrema, one does not need to know the type (loc. min, max, or saddle) of the cr. pts. See Note 2 on p. 17-7 as to why. Therefore,

one does not need the 2nd-derivative test when finding abs. min. & abs. max.

Ex. 3 Find the abs. extrema of $f(x,y) = xy^2$ on region B shown in the figure.



Sol'n:

Step 1 of Algorithm:

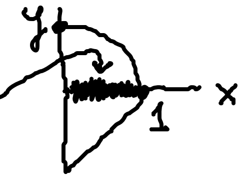
Find cr. pts. inside B.

$f = xy^2 \Rightarrow$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} y^2 = 0 \\ 2xy = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x \text{ is arbitrary.} \end{cases}$$

Thus, we have a segment of cr. pts rather than just one or two isolated points. (This is not typical and occurred only because I selected some simple $f(x,y)$ for this Example. My primary goal is to illustrate Step ② of the Algorithm.)

So, the critical pts. of $f(x,y)$ occupy an interval inside B:



Note: Whenever you search for cr. pts inside B, make sure that the answer you find is actually inside B, and discard all other answers.

Step ② of Algorithm:

Find the abs. max & min on the boundary. Consider each section of the boundary separately.

On ①: $x=0$, $-1 \le y \le 1$, $\boxed{f(x,y) = 0\cdot y^2 = 0}$

Note: In general, on a line $x = \text{const}$, $f(x,y) \equiv f(\text{const}, y)$ will be a function of y, and one will need to find its abs. extrema using the Calc. I Algorithm.

On ②: This is part of a circle $x^2+y^2=1$.

Recall: In this course we **use**

parametric, not Cartesian, eqs. of any circle!

1) Write the equation of the boundary:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad \boxed{0 \le t \le \pi/2} \Rightarrow f(x,y) = xy^2 = \boxed{\cos t \cdot \sin^2 t \equiv F(t)}$$

To find abs. extrema of $F(t)$ on $0 \le t \le \pi/2$, follow the Calc. I Algorithm.

2) Calc. I Algorithm for $F(t)$:

1. Find cr. pts. of $F(t)$ inside $[0, \pi/2]$.

$F'(t) = 0 \Rightarrow (\cos t \cdot \sin^2 t)' = 0 \Rightarrow$

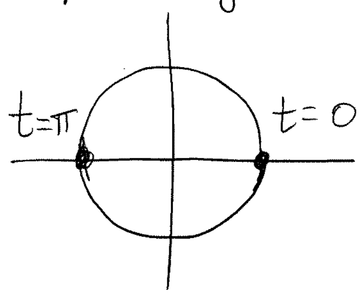
$-\sin t \cdot \sin^2 t + \cos t \cdot 2 \sin t \cdot \cos t = 0 \Rightarrow$

$\sin t \cdot (-\sin^2 t + 2\cos^2 t) = 0 \Rightarrow$

$$\left[\begin{array}{ll} \sin t = 0 & \text{(A)} \\ \quad \text{or} & \\ -\sin^2 t + 2\cos^2 t = 0 & \text{(B)} \end{array}\right.$$

Solve the trig. eqs. using the unit circle.

(A) $\sin t = 0$



$t = 0$ on boundary of $[0, \pi/2]$
Not inside $\Rightarrow$ discard.

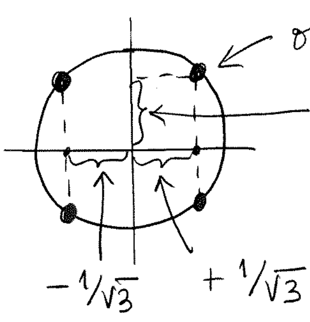
$t = \pi$ is not inside $[0, \pi/2] \Rightarrow$ discard.

(B) Need to solve for sin or cos first.
In this case the choice between $\sin t$ & $\cos t$ does not matter (in some HW problems it will!);

use $\boxed{\sin^2 t + \cos^2 t = 1} \Rightarrow \sin^2 t = 1 - \cos^2 t$

$\Rightarrow -\underbrace{\sin^2 t}_{1-\cos^2 t} + 2\cos^2 t = 0 \Rightarrow -(1-\cos^2 t) - 2\cos^2 t = 0$

$\Rightarrow -1 + 3\cos^2 t = 0 \Rightarrow \cos^2 t = 1/3 \Rightarrow \cos t = \pm 1/\sqrt{3}$



only this point is inside $[0, \pi/2]$

$\sin t = +\sqrt{1 - \cos^2 t} = \sqrt{1 - 1/3} = \sqrt{2/3}$

Thus, cr. pt. $(x = \underbrace{\sqrt{1/3}}_{\cos t_0},\ y = \underbrace{\sqrt{2/3}}_{\sin t_0})$

(17-12)

2. (= 2nd step of Calc. I Algorithm)

Choose the min. and max values among:

$\{ F(t_0) = f(\cos t_0, \sin t_0),\ F(0),\ F(\pi/2) \}$.

$F(t_0) = f(\sqrt{1/3}, \sqrt{2/3}) = \sqrt{1/3} \cdot (\sqrt{2/3})^2 = 2/(3\sqrt{3})$.

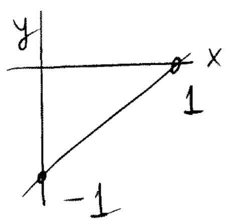
$F(0) = \cos 0 \cdot \cancel{\sin^2 0}^{\,0} = 0$

$F(\pi/2) = \cancel{\cos \pi/2}^{\,0} \cdot \sin^2 \pi/2 = 0$, $\Rightarrow$

$$\boxed{\begin{array}{l} \text{Abs. min. on } ② = 0 \quad @ (x,y) = (0,1) \ \& \ (1,0) \\ \text{Abs. max on } ② = 2/(3\sqrt{3}). \ @ (x,y) = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \end{array}}$$

On ③: Follow the steps we used on ②.

1) Write the equation of this section of the boundary:



$y = x - 1 \qquad 0 \le x \le 1 \quad \Rightarrow$

$f(x,y) = x \cdot (x-1)^2 = x^3 - 2x^2 + x \equiv F(x)$

2) Calc. I Algorithm for F(x)

1. Find cr. pts. of F(x) inside $0 \le x \le 1$.

$F'(x) = 0 \Rightarrow 3x^2 - 4x + 1 = 0$ ← quadratic eq.

$\Rightarrow (3x-1)(x-1) = 0 \Rightarrow$ (use Quadratic Formula or Inspection)

$\left[\begin{array}{l} 3x - 1 = 0 \\ x - 1 = 0 \end{array}\right.$ or $\Rightarrow \left[\begin{array}{l} x = 1/3 \\ x = 1 \end{array}\right.$.

$x = 1/3$ ← inside $[0,1]$; OK;

$x = 1$ ← not inside $[0,1]$ But at the end pt.,$\Rightarrow$ consider later.

Thus, the cr. pt. of F(x) inside $[0,1]$ is:

$x_0 = \frac{1}{3},\ y_0 = x_0 - 1 = -2/3$

2. Find the min & max among:

$$\{F(x_0),\ F(0),\ F(1)\}.$$

$F(x_0) \equiv F(1/3) = 1/3 \cdot (-2/3)^2 = 4/27$

$F(0) = 0 \cdot (0-1)^2 = 0$

$F(1) = 1 \cdot (1-1)^2 = 0 \quad \Rightarrow$

$$\boxed{\begin{array}{l} \text{Abs. min on } ③ = 0 \ (@ (x=0, y=-1) \text{ and } @ (x=1, y=0)) \\ \text{Abs. max on } ③ = 4/27 \ (@ (x=1/3, y=-2/3)). \end{array}}$$

Step 3 of Algorithm (Calc. III):

Find the min & max among:

- all cr. pts. inside B (Step 1) &
- abs. min & abs. max on boundary of B (Step 2).

$$\{\text{none}, \underbrace{0}_{\text{on } ①}, \underbrace{0,\ 2/(3\sqrt{3})}_{\text{on } ②}, \underbrace{0,\ 4/27}_{\text{on } ③}\},$$

Thus:

$$\boxed{\begin{array}{l} \text{Abs. min on } B = 0 \ (@ (x=0, -1 \le y \le 1) \ \& \ @ (x=1, y=0)) \\ \text{Abs. max on } B = 2/(3\sqrt{3}) \ (@ (1/\sqrt{3}, \sqrt{2/3})). \end{array}}$$ ✓

Note 1 : If in a HW problem, the boundary of B has several sections, as in Ex. 3, one can use a shortcut. Indeed, we double counted the corners $(x,y) = (0,-1), (0,1), (1,0)$:
E.g., $(0,-1)$ is on 1 and on 3. Therefore, in Step 2 one can:

- first, find the cr. pts. on all sections of B; away from the corners;
- second, compare the values of f @ these cr. pts. with the values of f @ the corners.

Then each corner is counted only once.

Note 2 If the boundary of B is a full circle, you do not need to check its corners: a circle has no corners and hence no end points. I.e., even though on a full circle, $0 \le t \le 2\pi$, one does not need to check $t=0$ & $t=2\pi$.