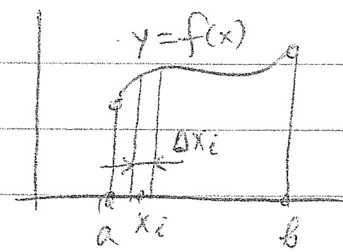
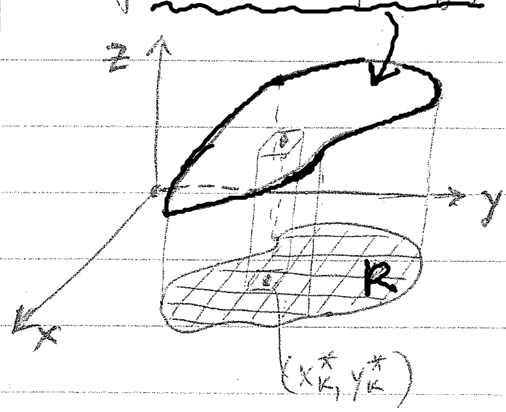
Sec. 15.1-A Double integrals as a volume.
(not the title of the section)

Recall from Calc. 1:

Area under $f(x)$ =
$\lim_{\max \Delta x_i \to 0} \sum f(x_i)\Delta x_i$ =
$= \int_a^b f(x)\,dx$.



Now: Find the volume enclosed between the region R in the xy-plane and the (positive) surface $z = f(x,y)$.



The procedure is similar to that used for $y = f(x)$. We subdivide the region R into small rectangles. Their sides are parallel to the coordinate axes.

2) Choose any point (x_k^*, y_k^*) inside the k-th rectangle. Construct the rectangular parallelepiped with the height $f(x_k^*, y_k^*)$. The volume of this parallelepiped approximates

the volume between the surface and the k-th rectangle:

$$V_k \approx f(x_k^*, y_k^*) \underbrace{\Delta x_k \Delta y_k}_{\Delta A_k} \quad (*)$$

3) As the size of the rectangles gets smaller:

See Fig. 8 in Sec. 15.1

i) the approximation of V_k by formula (*) becomes better;

ii) the rectangles get smaller and thus cover the region R more thoroughly.

iii) Also, it is intuitively clear that as the rectangles get smaller, the particular choice of the point (x_k^*, y_k^*) inside the rectangle gets less important, if the surface $f(x,y)$ is continuous.

Therefore, we define the required volume as

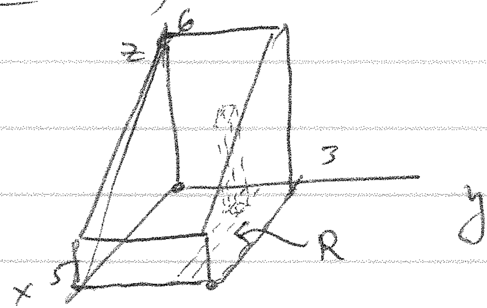
$$\boxed{V = \lim_{\max \Delta A_k \to 0} \sum f(x_k^*, y_k^*) \Delta A_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k \overset{\text{def}}{=} \iint_R f(x,y)\, dA}$$

Volume above region R in xy-plane and below surface $z = f(x,y)$.

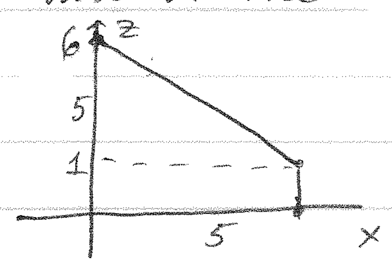
<u>Ex. 1</u> Evaluate the double integral by interpreting it as the volume of a solid.

$$\iint_R (6-x)\, dA, \quad R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 3\}.$$

<u>Sol'n:</u> 1) Make a sketch:



2) To find its volume, we notice that it is a prizm with the height along y-axis and the base in the xz-plane of the shape shown.



$$V = A_{base} \cdot \text{height}$$
$$= \left(\frac{5 \cdot 5}{2} + 1 \cdot 5\right) \cdot 3 = \frac{105}{2}.$$

<u>HW</u>: #11, 13.

Topic <u>Average value of $f(x,y)$ over R</u> – please Browse ~~pp.~~ 997, 998 (no HW about it; the book doesn't have, It is a 2D analogy of (Sec. 6.5), $\frac{1}{b-a}\int_a^b f(x)\,dx = f_{ave}$.